MATHEMATICS 435-EXAM 2(TAKE-HOME)-DUE 3/28/2002

1. Consider the Cauchy problem for the heat equation on the half-line $\{x > 0\}$, with Neumann boundary conditions:

$$u_t - u_{xx} = 0, \quad u = u(x,t), x > 0, t \geq 0$$
$$u_x(0,t) = 0$$
$$u(x,0) = f(x)$$

(i)The solution may be written in the form:

$$u(x,t) = \int_0^\infty q(t,x,y)f(y)dy.$$

Find an explicit expression for $q(t,x,y)$, depending only on the standard heat kernel $p(t,x) = \frac{1}{\sqrt{4\pi t}}\exp(-x^2/4t)$.

(ii)Find conditions on $f(x)$ which guarantee that the solution $u(x,t)$ converges to $f(x)$ as $t \to 0^+$, pointwise for every $x \geq 0$. *Prove* this convergence, either directly or by reducing to a known result on the equation on the whole line.

(iii)Under which conditions on f would one have *uniform* convergence to the initial data f as $t \to 0^+$? Justify.

2.(i) Find a formula for the solution of the Cauchy problem for the homogeneous heat equation on the half-line $x > 0$, with Neumann boundary conditions:

$$u_{tt} - u_{xx} = 0, \quad x > 0, t \geq 0$$
$$u_x(0,t) = 0$$
$$u(x,0) = 0, \quad u_t(x,0) = g(x)$$

(ii)The solution found in (i) may be regarded as an operator $\mathcal{S}(t)$, which assigns to the function g the solution at time t, $u(\cdot,t) = \mathcal{S}(t)[g]$. *Duhamel's principle* says the solution of the non-homogeneous problem with zero initial conditions and 'source term' $h(x,t)$ is given by:

$$v(\cdot,t) = \int_0^t \mathcal{S}(t-s)[h(\cdot,s)]ds.$$

Use Duhamel's principle to find an *explicit* formula for the solution of the following non-homogeneous problem for the wave-equation on the half-line $x > 0$, with Neumann boundary conditions:

$$u_{tt} - u_{xx} = h(x,t), \quad x > 0, t \geq 0$$
$$u_x(0,t) = 0$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

(iii)Now show that the formula you found in (i) does satisfy both the equation and the boundary condition.

3. Let $u(x,t)$ be the solution of the heat equation on the whole line, with initial data:

$$f(x) = x, x < -1; \quad f(x) = x + 2, -1 < x < 1, \quad f(x) = x, x > 1.$$

(i)Find an explicit formula for the solution, using only the special function:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} dp;$$

(ii)Use your solution to find the limits:

$$\lim_{t \to 0^+} u(-1,t), \quad \lim_{t \to 0^+} u(1,t).$$

Is the convergence of $u(\cdot,t)$ to f as $t \to 0^+$ uniform in x? Why, or why not?

(iii)Use your solution to prove that:

$$x < u(x,t) < x + 2,$$

for all $x \in \mathbb{R}$ and all $t > 0$.

(iv)Sketch the solution, for large t and for t close to 0.

4. Consider a solution of the wave equation $u_{tt} - u_{xx} = 0$ on $[0,\pi]$, with Dirichlet boundary conditions and initial data $u(x,0) = f(x), u_t(x,0) = g(x)$. The formal solution can be written in the form:

$$u(x,t) = \sum_{n \geq 1} b_n(t) u_n(x),$$

where u_n are the eigenfunctions.

The goal of the following 4 steps is to give a different proof of energy conservation. In particular, 'conservation of energy' *cannot be used* in their solution.

(i)Compute the energy $E_n(t)$ of the 'nth-harmonic', $b_n(t)u_n(x)$. Recall that the energy of a function $v(x,t)$ is defined as:

$$E_v(t) = \int_0^\pi \frac{1}{2}(v_x^2 + v_t^2) dx.$$

(ii)Using only the result of part (i), and the differential equation satisfied by the $b_n(t)$, show that $E_n(t)$ is constant in time;

(iii)Show that, for each t:

$$E_u(t) = \sum_{n \geq 1} E_n(t).$$

(Hint: orthogonality of the eigenfunctions)

(iv)Use Parseval's equality to show that, at $t = 0$:

$$\sum_{n \geq 1} E_n(0) = \frac{1}{2} \int_0^\pi (f_x^2 + g^2) dx.$$