1. Fourier series

1. Function spaces and norms.

A function $f(x)$ defined on $\mathbb{R}$ with values in $\mathbb{R}$ or $\mathbb{C}$ is of *class* C^k if it has k continuous derivatives (in particular, C^0 means continuous). The C^k *norm* of a C^k periodic function f is defined as:

$$||f||_{C^k} := \max_{x\in\mathbb{R}}(|f(x)| + |f'(x)| + \ldots |f^{(k)}(x)|).$$

(These numbers may be infinite for general C^k functions f; but they are always finite if f is periodic, the case of interest here). In particular, $||f||_{C^0} = \max_{x\in\mathbb{R}} |f(x)|$, and we have:

$$f_n \to f \text{ uniformly in } \mathbb{R} \iff ||f_n - f||_{C^0} \to 0.$$

A continuous function f defined on $\mathbb{R}$ is L^1 (respectively L^2) if

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty, \quad \text{resp.} \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty.$$

When this is the case, we define the L^1 and L^2 norms by:

$$||f||_{L^1} = \int_{-\infty}^{\infty} |f(x)|dx, \quad ||f||_{L^2} = (\int_{-\infty}^{\infty} |f(x)|^2 dx)^{1/2}.$$

Of course, a sequence f_n converges to f in L^2 if and only if $||f_n - f||_{L^2} \to 0$, and similarly for convergence in L^1.

Clearly C^k, L^1 and L^2 are all infinite-dimensional vector spaces (with real or complex scalars). The H^k norm ($k \geq 0$ an integer) , which combines the C^k and L^2 norms, is sometimes useful:

$$||f||_{H^k} = (||f||^2_{L^2} + ||f'||^2_{L^2} + \ldots + ||f^{(k)}||^2_{L^2})^{1/2}.$$

Properties.(a)All these norms have the following properties:

$$(i)||f|| \geq 0 \text{ for all } f, \text{ and is zero only for the zero function ;}$$
$$(ii)||cf|| = |c|||f|| \text{ for any constant } c \in \mathbb{R}(\text{ or } c \in \mathbb{C});$$
$$(iii)||f+g|| \leq ||f|| + ||g|| \text{ for any two functions } f, g.$$

In fact, these three properties define the term 'norm' on a vector space. One should thing of a norm as a measure of 'size' of a function. For vectors in $\mathbb{R}^k$ there is a 'natural' way to measure length; but for functions this is not the case, and the notion of 'size' that is most useful depends on the particular problem. The reason norms are useful is that they allow us to make quantitative statements (for example, stability statements for solutions of PDE).

(b)For 2π-periodic functions on $\mathbb{R}$, the following *inequalities* are easily seen to hold:

$$||f||_{L^2[-\pi,\pi]} \leq C||f||_{C^0}; \quad ||f||_{H^k[-\pi,\pi]} \leq C_k||f||_{C^k}.$$

Here when computing L^2 or H^k norms we compute the integrals only in $[-\pi, \pi]$. One may take $C = (2\pi)^{1/2}$ and $C_k = [(k+1)2\pi]^{1/2}$. But in truth the precise value of the constant is irrelevant for most purposes; all that matters is that it is independent of f. What these inequalities mean is that the C^k norms are 'finer' than the H^k norms (one instance of this is that the first inequality shows that uniform convergence implies convergence in L^2).

2. Fourier series of periodic functions.

The formal Fourier expansion and complex Fourier coefficients of a 2π-periodic function are defined by:

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

For $n \geq 1$, we define the partial sum and averaged partial sum (or 'Cesàro partial sum') by:

$$s_N(x) = \sum_{-N}^{N} c_n e^{inx}, \quad \sigma_N = \frac{1}{N+1} \sum_{j=0}^{N} s_j.$$

Integration by parts easily shows the following:

$$f \in C^1_{per} \Longrightarrow c_n[f'] = (-in)c_n[f];$$

inductively, we have for $f \in C^k_{per}$:

$$c_n[f^{(k)}] = (-in)^k c_n[f].$$

(The subscript *per* emphasizes we are looking at $2\pi-$periodic functions.) From the definition of the Fourier coefficients, it is obvious that we have:

$$|c_n| \leq \frac{1}{2\pi} ||f||_{L^1[-\pi,\pi]} \leq ||f||_{C^0}, \text{ for all } n \in \mathbb{Z}, f \in C^0_{per};$$

Combining this with the preceding fact, we see that for $f \in C^k_{per}$:

$$|n|^k |c_n[f]| = |c_n[f^{(k)}| \leq C||f^{(k)}||_{C^0} \leq C||f||_{C^k},$$

or equivalently:

$$|c_n| \leq \frac{C}{|n|^k} ||f||_{C^k}, \text{ for all } n \in \mathbb{Z}, n \neq 0, \text{ if } f \in C^k_{per}.$$

This is the easiest instance of the *basic principle*:

f has continous derivatives of higher order $\Longleftrightarrow$ faster decay of the Fourier coefficients as $|n| \to \infty$.

3. The first main theorem.

The space $\mathcal{P}_N$ of (complex) trigonometric polynomials of degree N is defined as: all linear combinations of $\{e^{-iNx}, e^{-i(N-1)x}, \ldots, e^{-ix}, 1, e^{ix}, \ldots, e^{i(N-1)x}, e^{iNx}\}$ with complex coefficients. It is a vector space with scalars $\mathbb{C}$, of dimension $2N+1$ (and incidentally a subspace of each of the spaces defined above!) Clearly for any periodic f the partial sums $s_N(x)$ and $\sigma_N(x)$ are elements of $\mathcal{P}_N$.

Theorem 1. Let $f \in C^0_{per}$. Then s_N is the best approximation of f by an element of $\mathcal{P}_N$, in the L^2 sense:

$$||f - s_N||_{L^2} = \min\{||f - g||_{L^2}; g \in \mathcal{P}_N\}.$$

The *proof* follows from the calculation ('completing the squares'): if $g = \sum_{-N}^{N} b_n e^{inx} \in \mathcal{P}_N$:

$$||f - g||^2_{L^2[-\pi,\pi]} = ||f||^2_{L^2[-\pi,\pi]} + 2\pi \sum_{-N}^{N} |c_n - b_n|^2 - 2\pi \sum_{-N}^{N} |c_n|^2.$$

Clearly the right-hand side is minimized by choosing $b_n = c_n$ for all n, which means $g = s_N$.
Setting $c_n = b_n$, we obtain:

$$||f - s_N||^2_{L^2[-\pi,\pi]} = ||f||^2_{L^2[-\pi,\pi]} - 2\pi \sum_{-N}^{N} |c_n|^2,$$

which implies *Bessel's inequality*:

$$\sum_{-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi} ||f||^2_{L^2[-\pi,\pi]},$$

as well as the fact that we have equality ('*Parseval's equality*') exactly when s_N converges to f in the L^2 sense. (Theorem 3 below shows this is always the case for continuous f.)

Now assume $f \in C^1_{per}$, and consider Bessel's inequality for the derivative f':

$$\sum_{-\infty}^{\infty} n^2 |c_n[f]|^2 = \sum_{-\infty}^{\infty} |c_n[f']|^2 \leq \frac{1}{2\pi} ||f'||_{L^2[-\pi,\pi]}.$$

Combining this with the Cauchy-Schwarz inequality, we obtain the very useful statement:

$$\sum_{-\infty}^{\infty} |c_n| \leq |c_0| + (\sum_{n\neq 0} \frac{1}{n^2})^{1/2} (\sum_{n\in\mathbb{Z}} n^2 |c_n|^2)^{1/2}$$
$$\leq C(||f||^2_{L^2[-\pi,\pi]} + ||f'||^2_{L^2[-\pi,\pi]})^{1/2} = C||f||_{H^1[-\pi,\pi]} \leq C'||f||_{C^1}.$$

This argument also works for higher derivatives to give a proof of the following.

Proposition 2. Let $f \in C^k_{per}$. The Fourier coefficients of f satisfy:

$$\sum_{-\infty}^{\infty} |n|^{k-1} |c_n| \leq C||f||_{H^k} \leq C'||f||_{C^k},$$

where C, C' are positive constants independent of f (but dependent on k).

This is another (and very useful) instance of the basic principle stated above.

4. The second main theorem.

Even though the partial sums $s_N(x)$ do not always converge to $f(x)$ (even pointwise!) if f is only assumed to be continuous, we do have *uniform* convergence of the Cesàro sums σ_N:

Theorem 3 (Fejér). If f is continuous and periodic, $\sigma_N \to f$ as $N \to \infty$, uniformly in $\mathbb{R}$.

Proof.(*outline*) Define:

$$K_N(x) = \frac{1}{N+1} \sum_{j=0}^{N} \sum_{n=-j}^{j} e^{inx} = \frac{1}{N+1} \sum_{j=-N}^{N} (N+1-|j|) e^{ijx}.$$

We have the expression for the averaged partial sum σ_N:

$$\sigma_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) K_N(x-y) dy.$$

It can be shown that $K_N(x)$ ('Fejér's kernel') admits the alternative expression:

$$K_N(x) = \frac{1}{N+1} \frac{\sin^2[(N+1)x/2]}{\sin^2(x/2)} (x \neq 0), \quad K_N(0) = N+1.$$

From this expression, it is clear that $K_N \geq 0$. From the definition of K_N, it follows easily that $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)dx = 1$. One can also show that, for each $0 < \delta < \pi$:

$$\lim_{N\to\infty} (\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}) K_N(x) dx = 0.$$

Given $\epsilon > 0$, we first fix $\delta > 0$ so that:

$$\max_{|y|<\delta} |f(x+y) - f(x)| < \epsilon/2, \text{ for each } x \in [-\pi, \pi].$$

(This is possible since f is uniformly continuous, given that $f \in C^0_{per}$.) Then take N sufficiently large, so that this last integral (which depends on δ) is smaller than $\epsilon/(2||f||_{C^0})$. Using the three properties of K_N just mentioned and a change of variables, we may write:

$$\begin{aligned}|\sigma_N(x) - f(x)| &= |\int_{-\pi}^{\pi} K_N(x-y)(f(y) - f(x)dy| = |\int_{-\pi}^{\pi} K_N(y)(f(x+y) - f(x))dy| \\ &\leq \int_{-\pi}^{\pi} K_N(y)|f(x+y) - f(x)|dy \\ &\leq \int_{-\delta}^{\delta}(\dots)dy + \int_{-\pi}^{-\delta}(\dots)dy + \int_{\delta}^{\pi}(\dots)dy \\ &\leq (\max_{|y|\leq\delta}|f(x+y) - f(y)|)\int_{-\pi}^{\pi} K_N(y)dy + 2||f||_{C^0}(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi})K_N(y)dy \\ &\leq \frac{\epsilon}{2} + 2||f||_{C^0}\frac{\epsilon}{2},\end{aligned}$$

from the choice of δ and N. Since ϵ is arbitrary, this concludes the proof.

Corollary 4. (*Weierstrass' theorem.*) Any continuous function in $[-\pi, \pi]$ can be uniformly approximated by trigonometric poynomials.

This clearly follows from the theorem, since $\sigma_N \in \mathcal{P}_N$.

Remark. Using Corollary 4 and Taylor's theorem, one can prove the following version of Weierstrass' theorem: any continuous function in a bounded interval $[a, b]$ can be approximated by polynomials, uniformly on $[a, b]$. (homework problem).

Corollary 5. If f is continuous and periodic, $s_N \to f$ in the L^2 sense.

This follows by combining the minimizing property (theorem 1), theorem 3 and the inequality between C^0 and L^2 norms:

$$||f - s_N||_{L^2[-\pi,\pi]} \leq ||f - \sigma_N||_{L^2[-\pi,\pi]} \leq \sqrt{2\pi}||f - \sigma_N||_{C^0} \to 0.$$

Corollary 6.(*Riemann-Lebesgue lemma*) If f is continuous and periodic, $c_n \to 0$ as $n \to \pm\infty$.

Given $\epsilon > 0$, let N be so large that $||f - \sigma_N||_{C^0} < \epsilon$. Then if $n > N$, the Fourier coefficient $c_n[\sigma_N] = 0$, so $|c_n[f]| = |c_n[f] - c_n[\sigma_N]| < \sqrt{2\pi}||f - \sigma_N|| < \epsilon/\sqrt{2\pi}$. Since ϵ is arbitrary, this proves the claim.

Corollary 7. (*Identity principle*) If f, g are continuous, periodic functions with the same Fourier coefficients, then $f = g$.

This is clear, since $\sigma_N[f] = \sigma_N[g]$ for each N, and hence:

$$||f - g||_{C^0} \leq ||\sigma_N[f] - f||_{C^0} + ||\sigma_N[g] - g||_{C_0} \to 0 \text{ as } N \to \infty.$$

Corollary 7 is used in the proof of the *third main theorem:*

Theorem 8 (uniform convergence). If $f \in C^1_{per}$, then $s_N \to f$ uniformly in $\mathbb{R}$. (In fact it is enough for f' to be piecewise continuous). More generally, if $f \in C^k_{per}$, $s_N \to f$ in C^{k-1} norm.

The fact that $\sum_{n\in\mathbb{Z}} |c_n| < \infty$ and the Weierstrass M-test easily imply that $s_N(x)$ converges uniformly to a function $g(x)$. This and the estimate:

$$|c_n[g] - c_n[s_N]| \leq 2\pi||g - s_N||_{C^0}$$

show that, for each $n \in \mathbb{Z}$, $c_n[s_N] \to c_n[g]$ as $N \to \infty$. Since, for each n, $c_n[f] = c_n[s_N]$ for $N > |n|$, this implies $c_n[f] = c_n[g]$ for all n. By corollary 7, $g = f$.