1. (Method of characteristics.) Consider the Cauchy problem for the first-order linear equation:

\[ xu_x + (x + y)u_y = 1, \quad u = u(x, y), \quad u(1, y) = y. \]

(i) Write down the associated system of ODEs and a parametrization of the initial curve;

(ii) Solve the system and invert the mapping \((s, t) \mapsto (x, y)\) to find the solution \(u(x, y)\). Include the domain of the solution.

2. (One-dimensional wave equation on the half-line.) Consider the problem on the half-line \(\{x > 0\}\), for \(t > 0\):

\[ u_{tt} - u_{xx} = 0, \quad u_x(0, t) = b(t), \quad u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \]

where \(f(x)\) and \(g(x)\) are defined only for \(x > 0\).

(i) Assume that \(f(x) \equiv 0\) and \(g(x) \equiv 0\). Let \(B(t)\) be the antiderivative of \(b(t)\) satisfying \(B(0) = 0\). (that is, \(B' = b\)). Verify that the function \(u(x, t)\) defined by:

\[ u(x, t) = -B(t - x) \text{ for } t > x; \quad u(x, t) = 0 \text{ for } x \geq t \]

solves the equation, and satisfies both the initial conditions (for \(x > 0\)) and the boundary condition at \(x = 0\).

(ii) Now assume \(b(t) \equiv 0\) and let \(f(x) = x^2\), \(g(x) = \cos(x)\). Write down an explicit expression for the solution, and explain (or verify directly) why it satisfies the boundary condition at \(x = 0\).

3. (Two-dimensional wave equation.) The representation formula for the solution of the Cauchy problem in \(\mathbb{R}^2\):

\[ u_{tt} - \Delta u = 0, \quad u = u(x, t), x \in \mathbb{R}^2, \quad u(x, 0) \equiv 0, u_t(x, 0) = g(x) \]

is:

\[ u(x, t) = \frac{1}{2\pi} \int_{B_t(x)} \frac{g(y)dA}{\sqrt{t^2 - |x - y|^2}}; \]

Note that the integral is taken over the full disk with center \(x\), radius \(t\): \(B_t(x) = \{y \in \mathbb{R}^2; |y - x| < t\}\).
(i) Assume that $g > 0$ exactly on the disk $\{|y| < R\}$ (and is zero elsewhere, where $|y|$ denotes distance from $y$ to the origin in $\mathbb{R}^2$). Fix a point $x$ in the plane and answer: for which values of $t$ is $u(x, t) \neq 0$? (There are two cases to consider, depending on $r = |x|$: $r \geq R$ or $r < R$. It helps to draw a diagram for each case.)

(ii) Plot your answer in a $t$ vs. $r$ graph ($t > 0$, $r > 0$.)

4. (Heat equation in an interval-Neumann BC) (i) Write down the solution for the initial-value problem in $[0, \pi]$:

$$u_t - u_{xx} = 0, \quad u_x(0, t) = u_x(\pi, t) = 0, \quad u(x, 0) = 1 + 5 \cos(x) + 7 \cos(3x).$$

(ii) Find the limits $\lim_{t \to \infty} u(x, t)$ and $\lim_{t \to 0^+} u(\pi, t)$. (Justify briefly.)

5. (Wave equation in an interval-Neumann BC) (i) Write down the formal solution to the non-homogeneous problem on $[0, \pi]$:

$$u_{tt} - u_{xx} = t, \quad x \in [0, \pi], \quad u(x, 0) = 0, \quad u_t(x, 0) = g(x), \quad u_x(0, t) = u_x(\pi, t) = 0$$

in terms of the Fourier cosine coefficients $a_n^g$ ($n \geq 0$) of $g$ in $[0, \pi]$ (note that the solution of the problem $v_{tt} - v_{xx} = t$ with zero initial conditions is $v(t, x) = t^3/6$.)

(ii) What condition on $g$ is needed for the solution in part (i) to be ‘classical’? (that is, $u(x, t)$ is $C^2$ in $(x, t)$.) Is the solution unique? (Answers may be based on d’Alembert’s formula.)