Answers and comments- Problems from Chapter 9.

9.5 This is a radial Cauchy problem in $\mathbb{R}^3$, so $v(r,t) = ru(r,t)$ solves the 1-dimensional wave equation on the half-line $r > 0$, with Cauchy data: $v(r,0) = 2r$, $v_t(r,0) = r + r^3$. These are odd functions of r (luckily), so to get $v(r,t)$ to vanish at $r = 0$, we blindly apply d'Alembert's formula as if the data were already defined on all of $\mathbb{R}$:

$$v(r,t) = \frac{1}{2}[2(r+t) + 2(r-t)] + \frac{1}{2}\int_{r-t}^{r+t} (s + s^3)ds.$$

This leads quickly to the answer:

$$u(r,t) = 2 + t(1 + r^2) + t^3.$$

9.6 Here again $u(r,0)$ and $u_t(r,0)$ are even functions of r, so we can assume they are defined on all of $\mathbb{R}$ and apply d'Alembert's formula directly. The answer is:

$$u(r,t) = e^{-(r^2+t^2)}[a\cosh(2rt) + (t + \frac{b}{2})\frac{\sinh(2rt)}{r}].$$

9.8

$$u_{mn}(x,y) = \cos(mx/a)\cos(ny/b), \quad \lambda_{mn} = \frac{m^2}{a^2} + \frac{n^2}{b^2}, \quad m,n = 0,1,2,\ldots$$

$$f(x,y) \sim A_0 + \sum_{m=1}^{\infty} A_m \cos(mx/a) + \sum_{n=1}^{\infty} \alpha_n \cos(ny/b) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\cos(mx/a)\cos(ny/b).$$

9.10 (a) Fix an integer $q > 2$, and let $n = q+1$. Obviously n and q have different parity, so $n^2 - q^2$ is odd, and setting $m = (n^2 - q^2 - 1)/2 \geq 1$, we have: $(m+1)^2 + q^2 = m^2 + n^2$. If $m = n$, it follows that $n^2 - 2n = q^2 + 1$, or $(n-1)^2 = q^2 + 2$, impossible since $n - 1 = q$. Hence $m \neq n$. Also, since $m + 1 = q^2 - q + 1$, $m + 1 = n = q + 1$ would imply $m = q$, or $q^2 = 2q$, impossible since $q > 2$. Hence $m + 1 \neq n$. Thus the linearly independent functions:

$$\sin(m+1)x \sin qy, \quad \sin mx \sin ny, \quad \sin nx \sin my$$

(we take the square $[0,\pi]^2$) all have the same eigenvalue. Since q can be taken arbitrarily large, this happens infinitely many times. (Of course, this argument is just a minor refinement of the remarks on p.247).

(b) If $\frac{m^2}{a^2}+\frac{n^2}{b^2}=\frac{p^2}{a^2}+\frac{q^2}{b^2}$ with $q\neq n$, it follows that $a^2/b^2=(m^2-p^2)/(q^2-n^2)$, which is a rational number.

9.11

$$2\pi[xJ_{n+1}(x)-nJ_n(x)+xJ_n'(x)]=\int_0^{2\pi}e^{ix\sin\theta}(xe^{-i\theta}-n+ix\sin\theta)e^{-in\theta}d\theta$$

$$=\int_0^{2\pi}e^{ix\sin\theta}(x\cos\theta-n)e^{-in\theta}d\theta$$

$$=-i\int_0^{2\pi}\frac{d}{d\theta}(e^{ix\sin\theta})e^{-in\theta}d\theta-n\int_0^{2\pi}e^{ix\sin\theta}e^{-in\theta}d\theta$$

$$=i\int_0^{2\pi}e^{ix\sin\theta}(-in)e^{-in\theta}d\theta-n\int_0^{2\pi}e^{ixsin\theta}e^{-in\theta}d\theta=0,$$

integrating by parts (there is no boundary term, since the functions are periodic.)