Answers to homework problems and comments- Ch.4

Two general principles: (i) D’Alembert’s formula for non-homogeneous problems involves an integration in spacetime, usually time-consuming. ‘Controlled trial-and-error’ (based on uniqueness) is often more efficient; (ii) In problems on the half-line, even-or odd- extensions of the initial data have to be carried out explicitly.

4.2 This is a problem on the half-line with \( u(0, t) \) prescribed, so one could use d’A’s formula with odd extensions. But note that the data is linear or quadratic. Any polynomial in \( x \) and \( t \) of the form:

\[
 u(x, t) = A(x^2 + t^2) + B\, xt
\]

automatically solves the wave equation with \( c = 1 \). Given \( u(x, 0) = x^2 \) and \( u_t(x, 0) = 6x \), \( A = 1 \) and \( B = 6 \) follow; and then (luckily) \( u(0, t) = t^2 \) also holds, so the answer is \( u(x, t) = x^2 + 6xt + t^2 \).

4.6 (a) By uniqueness, if the initial data are even functions of \( x \), so is the solution \( u(x, t) \), and then (assuming \( u \) is \( C^1 \) near the origin for each \( t \)) automatically \( u_x(0, t) = 0 \). Thus it suffices to extend \( f, g \) to \( \mathbb{R} \) as even functions, apply d’Alembert’s formula and restrict the result to the positive half-line (this just means we don’t care what happens for \( x < 0 \)). The conditions \( f'(0) = 0 \) and \( g'(0) = 0 \) (which are forced by the given boundary condition) imply the even extensions are \( C^2 \) (for \( f \)) and \( C^1 \) (for \( g \)), since the original \( f \) and \( g \) are \( C^2 \) (resp. \( C^1 \)) in the closed half-line. If these conditions hold, \( u(x, t) \) is a classical solution (without them, one could still consider the even extensions and run the machine, but the solution would have singularities- propagating along characteristics, as usual).

(b) Note that the even extension of \( f \) is \( f_e(x) = |x|^3 + x^6 \), that of \( g \) is:

\[
 g_e(x) = \sin^3 |x| .
\]

Thus the answer is:

\[
 u(x, t) = \frac{1}{2} (|x + t|^3 + |x - t|^3) + \frac{1}{2} ((x + t)^6 + (x - t)^6) + \frac{1}{2} \int_{x-t}^{x+t} \sin^3 |s| ds.
\]

4.12 Let \( v(x, t) = u(x, t) - \frac{4}{1+t} \). Then \( v \) solves:

\[
 v_{tt} - v_{xx} = \frac{2}{(1+t)^3}, \quad v(0, t) = 0, \quad v(x, 0) = 0, v_t(x, 0) = -1.
\]

We need to consider the odd extensions of the source term and the initial data for \( v \), as functions of \( x \). These are:

\[
 g_o(x) = -\text{sign}(x), \quad F_o(x, t) = 2\text{sign}(x)/(1+t)^3.
\]
Running the machine:

\[ v(x, t) = -\frac{1}{2}(|x + t| - |x - t|) + \int_0^t \frac{1}{(1 + \tau)^3} \left[ \int_{x-(t-\tau)}^{x+(t-\tau)} \text{sign}(\sigma) d\sigma \right] d\tau. \]

(Note the first term equals \(-t\) if \(x > t\) and equals \(-x\) if \(x < t\).)

To compute the integral in \(\sigma\), we use the fact (already used above) that \(|x|\) is an antiderivative of \(\text{sign}(x)\). Thus:

\[ \int_{x-(t-\tau)}^{x+(t-\tau)} \text{sign}(\sigma) d\sigma = |x + t - \tau| - |x - t + \tau|, \]

which is seen to equal: \(2(t - \tau)\) if \(\tau > t - x\) and \(2x\) if \(\tau < t - x\). So we have to consider two cases:

(i) \(x > t\). Then we always have \(\tau > t - x\) (since \(\tau > 0\)), hence:

\[ v(x, t) = -t + \int_0^t \frac{2(t - \tau)}{(\tau + 1)^3} d\tau = -t + (t - 1 + \frac{1}{t + 1}) = -\frac{t}{t + 1}. \]

(ii) \(x < t\). Then the integral in \(\tau\) must be split into two intervals, and we find:

\[ 2x \int_0^{t-x} \frac{d\tau}{(\tau + 1)^3} + \int_{t-x}^t \frac{2(t - \tau)}{(\tau + 1)^3} d\tau = x - \frac{1}{t - x + 1} + \frac{1}{t + 1}. \]

So in this case:

\[ v(x, t) = -x + \left( x - \frac{1}{t - x + 1} + \frac{1}{t + 1} \right). \]

We conclude the solution is:

\[ u(x, t) = 0 \text{ for } t < x, \quad u(x, t) = 1 - \frac{1}{t - x + 1} = \frac{t - x}{t - x + 1} \text{ for } t > x. \]

It is easy to check this function solves the equation (for \(x > 0\)), and satisfies the initial and boundary conditions (provided one uses the expression for \(x > t\) when checking \(u_t\) at \(t = 0\)).

**Remark 1.** Note the solution \(u\) is continuous on the line \(x = t\), but \(u_t\) is not (it has a jump of 2 across the line.) This corresponds to the fact that the given boundary condition \(u(0, t) = t/(1 + t)\) (which has derivative 1 at \(t = 0\)) is *incompatible* with the initial condition \(u_t(x, 0) = 0\) (which would give \(u_t(0, 0) = 0\) if \(u_t\) were continuous on the line \(x = t\)).
Also, once one realizes the solution for $x > t$ is a function of $t$ only, given the initial conditions it must be the zero function.

Part (b) is easy: the limit is 0 when $c \leq 1$, 1 when $c > 1$ (along the line $t = cx$, for $x \to \infty$).

**Remark 2.** Note this is a very interesting solution of the wave equation on the half line: it is driven entirely by the boundary condition (the rest of the data is zero), vanishes identically outside the light cone from the origin and is always non-zero inside that light cone. At any point $x > 0$, the perturbation at the boundary is first felt for $t > x$, and continues to be felt for all future time, asymptotically with the value 1. Note the solution has the form: $u(x, t) = 0$ for $x > t$, $u(x, t) = f(t - x)$ for $t > x$, where $f(t) = u(0, t)$. Can you show this is true in general? (Assume, say, $f(0) = f'(0) = 0$.)

4.14 Note that $e^x$ and $\sin t$ behave very simply under differentiating twice, so we easily see that $v(x, t) = u(x, t) + \frac{1}{4}e^x + \sin t$ solves the problem:

\[ v_{tt} - 4v_{xx} = 0, \quad v(x, 0) = \frac{1}{4}e^x, \quad v_t(x, 0) = 1 + \frac{1}{1 + x^2}. \]

Using the formula, we easily find the answer:

\[ u(x, t) = \frac{1}{4}e^x \cosh(2t) + t + \frac{1}{4}[\arctan(x + 2t) - \arctan(x - 2t)] - \frac{1}{4}e^x - \sin t. \]

4.16 Answer: $u(x, t) = e^x \sinh t + \frac{x^3}{6}$. (Straightforward application of the non-homogeneous formula.)

4.17(a) One way to solve this without the formula is to write the ‘source term’ (right-hand side) in the form $\cos x \cos t - \sin x \sin t$, and look for solutions of the form:

\[ v(x, t) = f(t) \cos x + g(t) \sin x \]

(the linear term $x$ can just be added at the end to get $u$); this leads to the ODE problems:

\[ f'' + f = \cos t, \quad f(0) = f'(0) = 0, \quad g'' + g = -\sin t, g(0) = 0, g'(0) = 1. \]

Elementary ODE methods lead to the solutions:

\[ f(t) = \frac{1}{2} t \sin t, \quad g(t) = \frac{1}{2} t \cos t + \frac{1}{2} \sin t. \]

This leads to the answer:

\[ u(x, t) = x + \frac{1}{2} \sin(t + x) + \frac{1}{2} \sin t \sin x. \]