**Answers to homework problems and comments- Ch.4**

*Two general principles*: (i)D'Alembert's formula for non-homogeneous problems involves an integration in spacetime, usually time-consuming. 'Controlled trial-and-error' (based on uniqueness) is often more efficient; (ii) In problems on the half-line, even-or odd- extensions of the initial data have to be carried out explicitly.

**4.2** This is a problem on the half-line with $u(0,t)$ prescribed, so one could use d'A's formula with odd extensions. But note that the data is linear or quadratic. Any polynomial in $x$ and $t$ of the form: $u(x,t) = A(x^2+t^2)+Bxt$ automatically solves the wave equation with $c = 1$. Given $u(x,0) = x^2$ and $u_t(x,0) = 6x$, $A = 1$ and $B = 6$ follow; and then (luckily) $u(0,t) = t^2$ also holds, so the answer is $u(x,t) = x^2 + 6xt + t^2$.

**4.6** (a) By uniqueness, if the initial data are *even* functions of $x$, so is the solution $u(x,t)$, and then (assuming $u$ is $C^1$ near the origin for each $t$) automatically $u_x(0,t) = 0$. Thus it suffices to extend $f, g$ to $\mathbb{R}$ as even functions, apply d'Alembert's formula and restrict the result to the positive half-line (this just means we don't care what happens for $x < 0$). The conditions $f'(0) = 0$ and $g'(0) = 0$ (which are *forced* by the given boundary condition) imply the even extensions are $C^2$ (for $f$) and $C^1$ (for $g$), since the original $f$ and $g$ are $C^2$ (resp. $C^1$) in the closed half-line. If these conditions hold, $u(x,t)$ is a classical solution (without them, one could still consider the even extensions and run the machine, but the solution would have singularities- propagating along characteristics, as usual).

(b) Note that the even extension of $f$ is $f_e(x) = |x|^3 + x^6$, that of $g$ is: $g_e(x) = \sin^3 |x|$. Thus the answer is:

$$u(x,t) = \frac{1}{2}(|x+t|^3 + |x-t|^3) + \frac{1}{2}((x+t)^6 + (x-t)^6) + \frac{1}{2}\int_{x-t}^{x+t} \sin^3 |s| ds.$$

**4.12** Let $v(x,t) = u(x,t) - \frac{t}{1+t}$. Then $v$ solves:

$$v_{tt} - v_{xx} = \frac{2}{(1+t)^3}, \quad v(0,t) = 0, \quad v(x,0) = 0, v_t(x,0) = -1.$$

We need to consider the *odd* extensions of the source term and the initial data for $v$, *as functions of* $x$. These are:

$$g_o(x) = -sign(x), \quad F_o(x,t) = 2sign(x)/(1+t)^3.$$

Running the machine:

$$v(x,t) = -\frac{1}{2}(|x+t| - |x-t|) + \int_0^t \frac{1}{(1+\tau)^3}[\int_{x-(t-\tau)}^{x+(t-\tau)} sign(\sigma)d\sigma]d\tau.$$

(Note the first term equals $-t$ if $x > t$ and equals $-x$ if $x < t$.)

To compute the integral in $\sigma$, we use the fact (already used above) that $|x|$ is an antiderivative of $sign(x)$. Thus:

$$\int_{x-(t-\tau)}^{x+(t-\tau)} sign(\sigma)d\sigma = |x+t-\tau| - |x-t+\tau|,$$

which is seen to equal: $2(t-\tau)$ if $\tau > t-x$ and $2x$ if $\tau < t-x$. So we have to consider two cases:

(i)$x > t$. Then we always have $\tau > t-x$ (since $\tau > 0$), hence:

$$v(x,t) = -t + \int_0^t \frac{2(t-\tau)}{(\tau+1)^3}d\tau = -t + (t - 1 + \frac{1}{t+1}) = -\frac{t}{t+1}.$$

(ii) $x < t$. Then the integral in $\tau$ must be split into two intervals, and we find:

$$2x\int_0^{t-x} \frac{d\tau}{(\tau+1)^3} + \int_{t-x}^t \frac{2(t-\tau)}{(\tau+1)^3}d\tau = x - \frac{1}{t-x+1} + \frac{1}{t+1}.$$

So in this case:

$$v(x,t) = -x + (x - \frac{1}{t-x+1} + \frac{1}{t+1}).$$

We conclude the solution is:

$$u(x,t) = 0 \text{ for } t < x, \quad u(x,t) = 1 - \frac{1}{t-x+1} = \frac{t-x}{t-x+1} \text{ for } t > x.$$

It is easy to check this function solves the equation (for $x > 0$), and satisfies the initial and boundary conditions (provided one uses the expression for $x > t$ when checking $u_t$ at $t = 0$).

*Remark 1.* Note the solution $u$ is continuous on the line $x = t$, but $u_t$ is not (it has a jump of 2 across the line.) This corresponds to the fact that the given boundary condition $u(0,t) = t/(1+t)$ (which has derivative 1 at $t = 0$) is *incompatible* with the initial condition $u_t(x,0) = 0$ (which would give $u_t(0,0) = 0$ if $u_t$ were continuous on the line $x = t$).

Also, once one realizes the solution for $x > t$ is a function of $t$ only, given the initial conditions it must be the zero function.

Part (b) is easy: the limit is 0 when $c \leq 1$, 1 when $c > 1$ (along the line $t = cx$, for $x \to \infty$).

*Remark 2.* Note this is a very interesting solution of the wave equation on the half line: it is driven entirely by the boundary condition (the rest of the data is zero), vanishes identically outside the light cone from the origin and is always non-zero inside that light cone. At any point $x > 0$, the perturbation at the boundary is first felt for $t > x$, and continues to be felt for all future time, asymptotically with the value 1. Note the solution has the form: $u(x,t) = 0$ for $x > t$, $u(x,t) = f(t-x)$ for $t > x$, where $f(t) = u(0,t)$. Can you show this is true in general? (Assume, say, $f(0) = f'(0) = 0$.)

**4.14** Note that $e^x$ and $\sin t$ behave very simply under differentiating twice, so we easily see that $v(x,t) = u(x,t) + \frac{1}{4}e^x + \sin t$ solves the problem:

$$v_{tt} - 4v_{xx} = 0, \quad v(x,0) = \frac{1}{4}e^x, \quad v_t(x,0) = 1 + \frac{1}{1+x^2}.$$

Using the formula, we easily find the answer:

$$u(x,t) = \frac{1}{4}e^x \cosh(2t) + t + \frac{1}{4}[\arctan(x+2t) - \arctan(x-2t)] - \frac{1}{4}e^x - \sin t.$$

**4.16** *Answer*: $u(x,t) = e^x \sinh t + \frac{xt^3}{6}$. (Straightforward application of the non-homogeneous formula.)

**4.17(a)** One way to solve this without the formula is to write the 'source term' (right-hand side) in the form $\cos x \cos t - \sin x \sin t$, and look for solutions of the form:

$$v(x,t) = f(t)\cos x + g(t)\sin x$$

(the linear term $x$ can just be added at the end to get $u$); this leads to the ODE problems:

$$f'' + f = \cos t, \quad f(0) = f'(0) = 0, \quad g'' + g = -\sin t, g(0) = 0, g'(0) = 1.$$

Elementary ODE methods lead to the solutions:

$$f(t) = \frac{1}{2}t\sin t, \quad g(t) = \frac{1}{2}t\cos t + \frac{1}{2}\sin t.$$

This leads to the answer:

$$u(x,t) = x + \frac{1}{2}\sin(t+x) + \frac{1}{2}\sin t \sin x.$$