Answers and comments- Problems from Chapter 2.

General comment: It is important to identify the natural *domain* of the solution. In general, this will be the largest *connected* open neighborhood of the (projection of) the initial curve on the (x, y) plane on which the solution is defined. More precisely, it is a neighborhood of the 'regular points' of the projected initial curve (those where the transversality condition holds). You should check this is true for all the domains found below.

2.2 ODE system: $x_t = x, y_t = x + y, z_t = 1$. Initial curve: $\Gamma_0(1, s, s)$. Solution: $x = e^t, y = se^t + te^t, z = t + s$. Inversion: $s = y/x - \ln x, t = \ln x$. Answer: $u(x, y) = \frac{y}{x}$ Domain: the half-plane $\{x > 0\}$ (connected neighborhood of the line $x = 1$).

2.4 ODE system: $x_t = y, y_t = -x, z_t = 0$. (*Remark:* In general, $z_t = 0$ means the solution is *constant along characteristics.* Projected initial curve: $(s, 0)$ ($s > 0$ in part (c)). Solution of system: $x = s\cos t, y = -s\sin t$. Inversion; $s = \sqrt{x^2 + y^2}, t = -\arctan(y/x)$. The characteristics $s = const.$ are arcs of circles. There are three cases:

(a)$u(x, 0) = x^2$. Then $u(x, y) = x^2 + y^2$ solves the equation and satisfies the initial condition for all $x \in \mathbb{R}$. The domain of u is all of $\mathbb{R}^2$ (the restriction $y > 0$ is not natural).

(b)$u(x, 0) = x$. Then $u(x, y) = \sqrt{x^2 + y^2}$ solves the equations and satisfies the initial condition only for $x > 0$; the domain is the half-plane $\{x > 0\}$ (note the origin is not a regular point, transversality fails there). There is no C^1 solution of the equation satisfying the initial condition for all $x \in \mathbb{R}$, for such a function would have to coincide with $\sqrt{x^2 + y^2}$ for $x > 0$ and with $-\sqrt{x^2 + y^2}$ for $x < 0$, in particular $u(x, 1) = sign(x)\sqrt{x^2 + 1}$ is not continuous at $x = 0$.

(c)$u(x, 0) = x, x > 0$. This is well-posed: the solution is $u(x, y) = \sqrt{x^2 + y^2}$, with domain the half-plane $\{x > 0\}$.

2.6 ODE system: $x_t = x, y_t = x^2 + y, z_t = 1 - (\frac{y}{x} - x)z$. Initial curve $\Gamma_0(1, s, 0)$. Solution: $x = e^t, y = e^{2t} + (s - 1)e^t, z = (1 - e^{-(s-1)t})/(s - 1)$ (if $s \neq 1$, $z = t$ (if $s = 1$). Inversion gives $t = \ln x, s = 1 + \frac{y}{x} - x$. This yields the solution (with domain the half-plane $\{x > 0\}$:

$$u(x, y) = \frac{x}{y - x^2}(1 - e^{(\frac{x^2 - y}{x})\ln x}), y \neq x^2, x > 0; \quad u(x, x^2) = \ln x, x > 0.$$

Remark: Since u is given by a different expression on the parabola $y = x^2$

(note the characteristics are parabolas $y = x^2 + (s-1)x$), it makes sense to ask whether u is C^1 in the whole half-plane. This is easier to check in (s,t) coordinates. Note $t = \ln x, s = 1 + \frac{y}{x} - x$ defines a diffeomorphism from the (s,t) plane to the $\{x > 0\}$ half-plane. In (s,t) coordinates, the question is whether the function defined by:

$$z(s,t) = \frac{1}{s-1}(1 - e^{-(s-1)t}), s \neq 1; \quad z(1,t) = t$$

is C^1 in (s,t). Continuity follows from the Taylor expansion at $s = 1$:

$$\frac{1}{s-1}(1 - e^{-(s-1)t}) = t + \frac{(s-1)}{2}t^2 + O((s-1)^2 + t^3).$$

With a little more work, one can show that z is C^1 in (s,t). For example, since $z_t = e^{-(s-1)t}$ for $s \neq 1$ and $z_t = 1$ for $s = 1$, we see that z_t is continuous everywhere. For z_s, compute z_s using the expression for $s \neq 1$ (and find the limit $-t^2/2$ as $s \to 1$), and directly using the definition when $s = 1$, to find again the value $-t^2/2$.

2.8 Letting $v = u^2/2$, the equation for v is: $xv_x + yv_y = 2v - 1$, with initial curve $\Gamma_0(s, s^2, s^6/2)$. The ODE system is $x_t = x, y_t = y, z_t = 2z - 1$, with solution $x = se^t, y = s^2e^t, z = \frac{1}{2}[1 + (s^2-1)e^{2t}]$, giving $v = (1/2)[1 - x^4y^{-2} + y^4x^{-2}]$ and the answer:

$$u(x,y) = (1 - x^4y^{-2} + y^4x^{-2})^{1/2}.$$

The domain of u is the set:

$$D = \{(x,y) \in \mathbb{R}^2 | x > 0, x^4y^{-2} - y^4x^{-2} < 1\}.$$

Note the inequality in the definition of D is satisfied on the arc of parabola $y = x^2$ $(x > 0)$ where the value of u is prescribed, so D is an open neighborhood of this curve (note the origin is excluded).

2.10 (a) Treating y as a parameter, the ODE system is $\tau_t = 1, x_t = 1 - y^2, z_t = 0$, with initial curve $\Gamma_0(s, 0, e^{y-s^2})$ and solution $\tau = t, x = s - (y^2-1)t, z = e^{y-s^2}$. Inverting we find $s = x + (y^2-1)\tau$, so the answer is:

$$u(x,y,\tau) = \exp[y - (x + (y^2-1)\tau)^2].$$

The domain is the whole open set $\{(x,y,\tau); (x,y) \in D, \tau > 0\}$.

(b) Sadly, the fish will die.

2.12 The ODE system is $x_t = z^2, y_t = 1, z_t = 0$, with initial curve $\Gamma_0(s, 0\sqrt{s}$ and solution $x = st + s, y = t, z = \sqrt{s}$. Inversion yields $s = x/(y+1), t = y$. So the answer is

$$u(x,y) = (\frac{x}{y+1})^{1/2},$$

with domain $\{(x,y)|x > 0, y > -1\}$.

2.16 This is straightforward: the ODE system is $x_t = x, y_t = y, z_t = -z$, with initial curve $\Gamma_0(\cos s, \sin s, 1)$ and solution $x = e^t \cos s, y = e^t \sin s, z = e^{-t}$. the characteristics are open rays from the origin. Inversion gives $e^t = \sqrt{x^2+y^2}$, so the answer is $u(x,y) = 1/\sqrt{x^2+y^2}$, with domain $\mathbb{R}^2 - \{(0,0)\}$.

2.24 ODE system: $x_t = x, y_t = -z, z_t = y$, initial curve $\Gamma_0(1, s, s)$, solution $x = e^t, y = s(\cos t - \sin t), z = s(\cos t + \sin t)$. inverting we find: $t = \ln x, s = y/(\cos(\ln x) - \sin(\ln x))$, so the answer is:

$$u(x,y) = y\frac{\cos(\ln x) + \sin(\ln x)}{\cos(\ln x) - \sin(\ln x)}.$$

The domain is the open vertical strip:

$$D = \{(x,y)|e^{-3\pi/4} < x < e^{\pi/4}\}.$$

2.26 Standard: the system is $x_t = -y, y_t = x, z_t = -z$, with initial curve $\Gamma_0(s, 0, 1)$ and solution $x = s\cos t, y = s\sin t, z = e^{-t}$. Characteristics are circles centered at the origin. Inversion gives $t = \arctan(y/x)$, so the answer is:

$$u(x,y) = \exp(-\arctan(y/x)),$$

with domain the half-plane $\{x > 0\}$.