MATH 435, FINAL EXAM -December 6, 2007. Closed book, closed notes, but you may use the handout 'Eigenfunctions, Green's functions, Poisson kernels'. The exam consists of **SIX** problems. Justify your answers (for credit.) *Time given:* 2 hours (7:15-9:15).

1.[18] Consider $f(x) = |x - \pi/2|$ in $[0, \pi]$. Let $s_N(x)$ be the sum of the first N terms of its Fourier *cosine* series.

(i) Sketch the graphs of f and f' and their extensions to $[-\pi, \pi]$ (f as an even function, f' as an odd function).

(ii)Does s_N converge to f uniformly in $[0, \pi]$? Explain.

(iii)Let $\sigma_N(x)$ be the partial sum of the Fourier *sine* series of f'. Find the pointwise limit $g(x)$ of $\sigma_N(x)$, for each x in $[0, \pi]$. Does σ_N converge to g uniformly in $[0, \pi]$? (Justify).

2.[16] (i) State the *maximum principle* for the *heat equation* in a bounded domain.

(If you don't remember it, stating the maximum principle for Laplace's equation is acceptable- but it won't help with part (ii).)

(ii) Let $u(r, \theta, t)$, $t \geq 0$, be the solution to the heat equation on the unit disk in $\mathbb{R}^2$, with boundary and initial conditions:

$$u(1, \theta, t) = \sin\theta, \quad u(r, \theta, 0) = r(2 - r)\sin\theta.$$

Find $M > 0$ (as small as possible) so that:

$$|u(r, \theta, t)| \leq M \text{ for all } t \geq 0.$$

3.[16] Solve the initial/boundary value problem for the **HEAT** equation in the square $[0, \pi] \times [0, \pi]$, $u = u(x, y, t)$:

$$u_t - \Delta u = 0, \quad u(x, 0, t) = 0, u(x, \pi, t) = 1, \quad u_x(0, y, t) = u_x(\pi, y, t) = 0,$$

$$u(x, y, 0) = \frac{y}{\pi} + \cos x \sin 2y.$$

(*Hint:* Note that $v(x, y) = \frac{y}{\pi}$ is harmonic and independent of t.)

4.[16](i) Write down an expression for the formal solution of the Dirichlet problem for the **WAVE** equation on the unit ball in $\mathbb{R}^3$ with **radial** initial conditions:

$$u_{tt} - \Delta u = 0, \quad u = u(r,t), \quad u(1,t) = 0.$$

$$u(r,0) = 0, \quad u_t(r,0) = g(r).$$

Show how the coefficients are computed from $g(r)$.

(ii) Find the solution $u(r,t)$ explicitly in case $g(r)$ is given by:

$$g(r) = \frac{\sin \pi r}{r}, r \neq 0; \quad g(0) = \pi.$$

(**Use the eigenfunctions given in the handout!**)

5.[16](i) Find the *most general solution* to the *exterior* Neumann problem for the unit disk:

$$\Delta u = 0 \text{ in } \{r > 1\}; \quad u_r(1,\theta) = \sin 2\theta.$$

(There will be some arbitrary constants in your answer.)

(ii) Which condition do you need to add to the problem to guarantee a unique solution? Explain why this condition would be sufficient.

6.[18] *The three items in this question are independent.*

You have a handout with expressions for Green's function $G_U(x;y)$ and the Poisson kernel $P_U(x;y)$ for the upper half-plane $U = \{(y_1, y_2); y_2 > 0\}$.

(i) State the *definition* of 'Green's function for a domain $D \subset \mathbb{R}^2$.' (Recall there are three defining conditions.)

(ii) Verify (directly, using the expression in the handout) that we have:

$$G_U(x;y) = G_U(y;x), \text{ for all } x, y \in U.$$

(iii) Verify that the Poisson kernel defines, for fixed $y \in \partial U$, a harmonic function of $x \in U$. It is enough to do this for the case $y = 0$, that is, to verify the statement:

$$\Delta\left(\frac{x_2}{x_1^2 + x_2^2}\right) = 0.$$

(Do it by direct differentiation, or use the expression of Δ in polar coordinates).