Problems on boundary-value problems, Green's functions.

1. Solve $\Delta u = 0$ in the exterior $\{r > a\}$ of a disk, with boundary condition $u = 1 + 3\sin\theta$ on $r = a$, with u *bounded.*

2. Derive Poisson's formula for the exterior of a disk:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - a^2}{a^2 - 2ar\cos(\theta-\tau) + r^2} h(\tau)d\tau.$$

($h(\tau)$ is the function giving the boundary values of u on the circle of radius a.)

3. Find the steady-state temperature distribution inside an annular plate $\{1 < r < 2\}$, assuming the outer edge is insulated, and on the inner edge the temperature is maintained at $\sin^2\theta$.

4. Find the harmonic function u in the half-disk $\{r < 1, 0 < \theta < \pi\}$ vanishing on the diameter ($\theta = 0, \pi$) and satisfying on the part of the boundary $r = 1$:

$$u = \pi\sin\theta - \sin 2\theta.$$

5. Prove uniqueness for the 'Robin problem':

$$\Delta u = f \text{ in D}\ , \quad \frac{\partial u}{\partial n} + au = g \quad \text{ on } \partial D,$$

where D is a bounded domain and a is a positive constant. (f and g are given functions on D and ∂D, respectively).

6. Let $\pi(x)$ be any C^2 function defined on all of three-dimensional space that vanishes outside some sphere. Show that:

$$\phi(x) = \int_{\mathbb{R}^3} G(x;y)\Delta\phi(y)dy,$$

where $G(x;y)$ denotes Green's function for $\mathbb{R}^3$ with pole at y.

7. Use the integral representation formula for the solution of Dirichlet's problem in the upper half-space to show:
(i) $u(x,y,z) \to 0$ as $z \to \infty$;
(ii) $u(x,y,z) \to h(x,y)$ as $z \to 0$ (*Hint:* Change variables, $s^2 = [(x - x_0)^2 + (y-y_0)^2]/z_0^2$, use $\int_0^\infty s(s^2+1)^{-3/2}ds = 1$.)

8.(i) Find Green's function for the upper half-plane $\{y > 0\}$ (ii) Use it to find the Poisson representation formula for the upper half-plane. (iii) and (iv) Repeat problem 7. for the upper half-plane.

9. Find the potential of the electrostatic field due to a point charge located outside a grounded sphere (this is Green's function for the exterior of the sphere; use the reflection method).

10. Find Green's function for the half-ball $D = \{r^2 < a^2, z > 0\} \subset \mathbb{R}^3$. (Use the solution for the whole ball and reflect it across the plane). Can you do it for the half-disk in the plane?

11. The function xy is harmonic in the half-plane $\{y > 0\}$ and vanishes on the boundary $\{y = 0\}$. The constant function 0 has the same property. Does this mean solutions are not unique?

Source: *W. Strauss, Partial Differential Equations: An Introduction*