

MATHEMATICS IN ANCIENT GREECE

Periods in Greek history [AG]. *Archaic Period: 750 BC to 490 BC* (Emergence of city-states to Battle of Marathon) The *Iliad* and the *Odyssey* were composed c. 750-720 (based on the `Trojan war' of 1250-1225 BC.) Between 670-500 BC, many city-states were ruled by tyrants (a word borrowed into the Greek language from Asia Minor, to signify a man who seizes control of the state by a coup and governs illegally.) *Classical Period: 490 to 323 BC* (Battle of Marathon to death of Alexander). The years from 480 (Persians driven from Greece) to 430 saw the rise of Athenian democracy (under *Pericles*). The *Parthenon* was built, 447-432. Playwrights *Sophocles*, *Aristophanes*, *Euripides* were active in Athens (c. 430-400). Following undeclared war between Athens and Sparta from 460-445, the *Peloponnesian war* (431-404; account by *Thucydides*) kept these two states (and their allies throughout the region) busy. This led to the ascendance of Sparta and the `rule of thirty tyrants' in Athens (404-403). *Socrates* was tried and executed in 399. His follower *Plato* wrote his *dialogues* (399-347) and founded the Academy. The accession of *Philip II* of Macedon (359) signals the beginning of a period of *Macedonian rule*. Philip invaded Asia in 336, and was assassinated the same year. His successor *Alexander III* (`the Great', 356-323 BC) continued the program of world conquest over the next 13 years, invading India in 325 BC. *Hellenistic Period (323-30 BC)*. Science, mathematics (*Euclid*) and culture flourished in Egypt (Alexandria) under the Ptolemaic dynasty. The *Museum* was built in Alexandria, 307-283. In 216 BC, Syracuse (in Sicily) became an ally of Carthage during the second Punic war (Rome vs. Carthage); in the siege of 212 BC, *Archimedes* was killed. In 31 BC, the Roman emperor Octavian defeated Antony and the Ptolemaic queen Cleopatra VII. With their suicide in 30BC, Rome annexed Egypt, the last Hellenistic kingdom to lose its independence.

[AG]: Burstein, S., Donlan, W., Pomeroy, S., Roberts, J., *Ancient Greece: A Political, Social and Cultural History*, Oxford U.P. (1999).

1- PYTHAGORAS AND HIS SCHOOL

Thales of Miletus (c. 624- 547 BC), a mathematician and astronomer, is one of the earliest Greek scientists we know of ([AG], pp. 122-123.) Among others, the following discoveries are attributed to Thales: (1) angles at the base of an isosceles triangle are equal; (2) the angle on a semicircle is a right angle (an ox was sacrificed to celebrate this one). As the first Greek astronomer, he observed the inequality of the lengths of the four seasons, and predicted a solar eclipse in 585 BC (conjecturally using the period of 223 lunations discovered by the Babylonians.)

A Thales story [Heath p. 89]: He once fell into a well when stargazing, and was rescued by `a clever and pretty maidservant from Thrace' (Plato), who admonished him for being so eager to know what goes on in the heavens, he couldn't see what was right in front of him.

Pythagoras was born in Samos (c. 572 BC), but to escape the rule of Polycrates (tyrant of Samos), moved to southern Italy (c. 531 BC), and founded a society at Croton.

‘The original Pythagoreans and their successors followed strict rules in their daily lives. Women were included in the Pythagorean communities and were imbued with philosophical principles that regulated the conduct of daily life, including many food taboos. They were strict vegetarians, for they believed in the transmigration of the soul.’ [AG p.123]

‘The Pythagoreans believed arithmetic held the key to understanding the universe. They postulated the earth was a sphere in the center of a system of hollow spheres holding the planets, which rotated from west to east at various rates. Their movement created a sound, but since the sound is always with us, we are unable to hear it’ [AG p. 123]. ‘It seems certain Pythagoras himself discovered that musical harmonies depend on numerical ratios: the octave representing 2:1 (string lengths at same tension), the fifth 3:2 and the fourth 4:3.’ [Heath p. 37]

Mathematical results attributed to the Pythagoreans [Heath, pp. 92-108] : (1) The sum of the angles of any triangle is equal to two right angles; (2) Pythagoras’ theorem, and construction of ‘Pythagorean triples’; (3) geometrical algebra, including solution of equations of degree 2; (4) irrational numbers (reportedly, the first of the Pythagoreans to make this discovery public was banished from the community.) (5) the five regular solids.

Math: (1) Proof of Pythagoras’ theorem, and its converse;. (2) Generation of Pythagorean triples; (3) An example of ‘geometrical algebra’[Heath p.100]

2- PLATO AND EUDOXUS

Plato (427-347), a philosopher who lived and taught in Athens in the period between the Peloponnesian War (431-404 BC) and the rise of Phillip of Macedon (360 BC), incorporated into his *dialogues* elements of a philosophy of mathematics (e.g. in the *Republic* and in *Meno*) [Heath p. 173-4], and students at his Academy (‘let no one ignorant of geometry enter here’) included important mathematicians, foremost among them Eudoxus of Cnidos (c.408-355 BC).

An important role in Platonic philosophy was played by ideal Forms [AG, p.356]: “A ring or a diadem or the perimeter of a shield might seem to the casual observer to be circles, but they are not circles in the same sense as the locus of all points in a plane equidistant from a given point. Some people might say that these concrete objects are real circles, while the geometrical object is imaginary, but for Plato only the concept is real. The tangible objects are debased copies, feeble imitations of the ideal Form. Plato believed in an opposition between the physical world of appearances, which are deceptive, and the intellectual universe of ideas, which represent reality.”

Concrete mathematics in Plato [Heath, p.175 ff]: in *Timaeus*, a description of the five regular (‘Platonic’) solids from their faces. Plato ascribes them to the four elements: the

tetrahedron to fire, the octahedron to air, the icosahedron to water and the cube to earth. The dodecahedron? With no elements left, he assigned it to the universe itself [Dunham2, 78-9]. Plato is also credited with a formula to generate *Pythagorean triples*.

Math:1- Compare Plato's rule with the Pythagoreans', and with the general solution in *Elements X* [Heath 47-8].

2-Classification of the regular solids (*Elements XIII*) [Dunham 2, p.80]

Eudoxus of Cnidos (c.408-c.355 BC) is considered by many one of the most original Greek mathematicians, second only to Archimedes. At the age of 23 he found his way to Athens and the Academy; being poor, he lived in Piraeus and 'trudged to Athens and back on foot each day'[Heath, p.187] . Returning to Cnidos, he then went to Egypt for 16 months, learning astronomy from the priests at Heliopolis, and making his own observations. Later he had his own observatory built at Cnidos (shades of Gauss...)

In astronomy, Eudoxus developed a detailed theory to explain the motion of the sun, moon, and the four planets by postulating concentric spheres (with earth at the center), connected in intricate ways. In mathematics, he is known for developing a theory of proportions allowing 'incommensurates', and for the 'method of exhaustions', which led to rigorous arguments for areas and volumes while avoiding infinitesimal quantities (and hence Zeno's paradoxes). Specifically, the proofs in the *Elements* of the formulas for the volume of a cone, and of the statement that areas of disks (resp. volumes of balls) are proportional to the squares (resp. cubes) of their radii are credited to him. Thus he circumvented the two deepest conceptual barriers to progress in mathematics at the time.

Math. 1-Proportions: definition of $a:b=c:d$ in *Elements V* [Struik, p.44]

2- Zeno's paradoxes: [Struik p. 43, Heath p. 192] *Dichotomy* and *Achilles* deny the possibility of motion on the assumption that magnitudes can be divided indefinitely. *Arrow* and *Stadium*: starting from the assumption that space and time are not indivisible, but rather composed of indivisible 'atoms', one again concludes motion is impossible.

3-Method of exhaustion- proof of the formula for the volume of a cone, by double contradiction [Struik p.46]

4- *Archimedean axiom*: given any magnitude $a > 0$ and any another magnitude $b > a$, adding a to itself enough times, the result (n times a) will be eventually greater than b . This is what made the 'exhaustion method' work. (Example later: quadrature of a parabola- Archimedes)

Menaechmus, a student of Eudoxus, was the first to use conic sections in the solution of a problem; namely, the problem of 'doubling the cube' ('solution' here means: find a geometric construction). This was the first time conics appeared 'in print'. The Greeks (e.g. Pappus) classified problems into (i) *plane* problems (those requiring only the line and the circle for their solution; (ii) *solid* (those requiring conic sections; (iii) *linear* problems, those requiring curves of higher order than conics (spirals, conchoids, cissoids, etc.) [Heath p. 139] (This means 'hard', so today we would call them 'nonlinear'.)

Math- 1- Reduction of the problem to the 'problem of two mean proportions' (Hippocrates, c.450 BC); solution as the intersection of two parabolas [Heath p. 160].
2- The two other 'challenge problems' of Antiquity were: (i) 'squaring the circle', i.e. constructing the area of a given disk using only straightedge and compass (Lindemann 1882 proved this is not possible- π is transcendental.) (ii) trisecting an arbitrary angle- this can be solved (*Pappus*) using the intersection of an ellipse and a hyperbola [Heath p.140].

3- EUCLID AND THE *ELEMENTS*

Euclid (and the Greek mathematicians who came after him) belongs to the *Hellenistic period*, which spans the time from the death of Alexander the Great (323 BC) to the death of Cleopatra VII in Egypt (30 BC). Following Alexander's death, the regent Perdiccas assigned parts of his empire to his generals (Alexander's son was born after his death, and eventually murdered before he could reign); in particular Egypt to Ptolemy, giving rise to the Ptolemaic dynasty. The first three Ptolemies transformed Alexandria (established at the mouth of the Nile by Alexander in 332 BC, and the site of his tomb) into the foremost city of the Hellenistic world. 'A liberal immigration policy created a multiethnic population, including Macedonians, Greeks, Egyptians and a vibrant Jewish community' [AG, p.455]... 'With the enormous wealth of Egypt at their disposal, the Ptolemies could afford to recruit and subsidize intellectuals, and to encourage artistic and scientific work by establishing a new type of institution'. The principal one was the research center known as the *Museum* (dedicated to the nine Muses), where scholars could pursue their studies unhindered and with full government support. To assist the Museum's scholars, Ptolemy I (r. 306-283 BC) established a library intended to contain every book written in Greek (visitors to Egypt were searched for books, which were seized if the library did not already have them in its collection- the owner was given a cheap copy) [AG p.457].

Around 300 BC, a man named Euclid was recruited to establish a School of Mathematics in Alexandria. Little is known about his life before then; we don't know his birthplace, or the dates of his birth and death. Presumably he received his training at the Academy, under Plato's followers (and got good letters of recommendation). Here is a Euclid story [Heath p. 203]: someone who had begun to study geometry with Euclid had no sooner learned his first theorem than he asked 'What shall I get for learning these things?' whereupon Euclid summoned his slave and said 'give this student some change, since he needs to make some gain out of what he learns'.

Euclid's *ELEMENTS*

(Source: [Dunham 2, pp. 31-81], [Heath pp. 216-253])

Quoting [Heath, p. 204]: 'This wonderful book, notwithstanding its imperfections, remains the greatest elementary textbook in mathematics the world is privileged to possess [most of K-12 geometry is based on it, often literally]. Scarcely any other book other than the Bible has circulated the world more widely, or been more edited and studied'.

The *Elements* contain an axiomatic development of essentially all the mathematics known at the time, deriving them from five 'postulates' (axioms), following strict logical rules. According to Heath, there is no preamble; Euclid plunges right into Definition 1. There are 23 definitions, 5 postulates and 5 'common notions' (rules of inference, so to speak).

Definition 1: A *point* is that which has no part.

Definition 2: A *line* is breadthless length.

Definition 4: A *straight line* is a line which lies evenly with the points on itself.

Definition 10: When a straight line standing on another straight line makes equal adjacent angles, each of the equal angles is *right*, and the two lines are *perpendicular*.

Definition 23: *Parallel* straight lines are lines in the same plane which, when produced indefinitely in both directions, do not meet one another in either direction.

POSTULATE 1: To draw a straight line from any point to any other point [is possible].

POSTULATE 2: To produce a line segment continuously in a straight line.

POSTULATE 3: To draw a circle with any given center and radius.

(constructions with straightedge and a 'collapsible compass')

POSTULATE 4: All right angles are equal to one another.

So far, so good. And then:

POSTULATE 5: If a straight line falling on two straight lines makes interior angles on the same side adding up to less than two right angles, the two straight lines, if produced indefinitely, will meet on that side of the intersecting line.

This sounds more like a theorem than an axiom, and indeed over the centuries people tried to 'prove it' from the other four. (That's a long story!) Euclid himself didn't like it- it is not used in the proof of his first 28 Propositions.

Remark- statements equivalent to Postulate 5: (i) through a point not on a line, one may draw a unique line parallel to the given line; (ii) the sum of the angles of a triangle is two right angles.

Common notion 1. Things which are equal to the same thing are equal to one another.

Common notion 2. If equals are added to equals, the wholes are equal.

Common notion 3. If equals are subtracted from equals, the remainders are equal.

Common notion 4. Things which coincide with one another are equal to one another.

(Interpretation: if a figure can be moved rigidly on the plane and exactly superimposed on another, they are equal- equal angles, sides, etc. This is more like a postulate [Dunham].)

Common notion 5. The whole is greater than the part.

Structure: There are 13 books, with a total of 465 propositions. Book I (48 propositions) deals with plane geometry (triangles, parallelism, parallelograms). Proposition 47 is Pythagoras' theorem, and Proposition 48 is its converse.

Example: Euclid's proof of Pythagoras' theorem (and the converse) [Dunham 2, p.48-52]

Book II (13 propositions) deals mainly with 'geometric algebra' (geometric proofs of algebraic identities, including the solution of quadratic equations). Prop. 12/13 is the 'law of cosines'.

Example: the square of $a+b$.(II.4)

Book III (37 propositions) deals with circles- properties of chords and central angles, intersecting circles, tangents, etc.

Example: opposite angles of an inscribed quadrilateral add up to two right angles.(III.22)

Book IV (16 props.) deals with inscribing/circumscribing regular polygons in a circle (with straightedge and compass, of course). The cases of 3, 4, 5, 6, 15 sides are dealt with.

Example: (IV.16) for the 15-gon, use $1/3-1/5=2/15$ [Dunham 2]

Remark 1: If you can do n sides, you can do $2n$. But 7, 9, 11, 17 sides are not covered- C.F. Gauss (in 1796, as a teenager) not only discovered how to construct the 17-gon, but also (later) characterized exactly which regular polygons are 'constructible'. (The others given above are *not*.)

Remark 2: The Pythagoreans were already familiar with the contents of Book IV, as well as Books I and II, and most of Book VI. The contents of Book II were known to and used by Hippocrates [Heath p. 131].

Book V (25 props.) presents the theory of proportions for 'incommensurate quantities', due to Eudoxus. (Recall the definition of $a:b=c:d$ given in [Struik, p.44].)

Example: (V.19) $a:b=c:d$ implies $(a-c):(b-d)=a:b$.

Book VI (31 props.) applies the theory of proportions from Book V to geometric problems- that is, it develops the theory of *similarity*.

Example (VI.8) If in a right-angled triangle a perpendicular is drawn from the right angle to the opposite side, the triangles adjoining the perpendicular are similar both to the whole triangle and to each other. (This gives a geometric construction of the geometric mean, or 'mean proportional').

Books VII (39), VIII (27) and IX (36) deal not with geometry, but with number theory.

Examples:

VII.1-3: 'Euclidean algorithm' to find the g.c.d. of two given numbers.

VII. 30: If a prime number divides the product of two numbers, it divides one of the two.

VII. 32: Any number is either a prime, or divisible by a prime number.

Book VIII is mostly about geometric progressions.

IX.14: uniqueness of prime factorization.

IX.20: there exist infinitely many prime numbers.

IX.35: sum of n terms of a geometric progression

IX.36: For any n , if the sum of the first $n+1$ powers of 2 (starting with 1) is a prime, then the product of that sum with the last power is a 'perfect number'.

Math: proofs of IX.14, IX.20, IX.36

Remark: L. Euler proved that all even perfect numbers have this form. Nobody knows if odd perfect numbers exist [Dunham 2 p. 82].

Book X (112) is algebra- there is a classification of certain irrational numbers obtained as sums or differences of 'incommensurate square roots.' Most of it is arcane, but the very first one contains the basis of the 'method of exhaustions':

X.1: If from any magnitude you subtract more than its half (or its half), from the remainder again more than its half (or its half), and so on continually, eventually there will remain a magnitude less than any pre-assigned magnitude.

Books XI (37), XII (18) and XIII (18) are about solid geometry.

Examples:

XI.21: The sum of the plane angles converging at a 'solid angle' is less than four right angles.

XII.2: Disks are to one another as the squares on their diameters.

XII.10: A cone is the same part of a cylinder with equal base and height.

XII.18: Spheres are to one another as the triplicate ratio of their diameters.

Book XII relies primarily on the 'method of exhaustion' (see discussion of Archimedes' work, below.)

The final Book XIII shows how to construct, and how to inscribe in a sphere, each of the five Platonic solids. XIII.18 arranges in increasing order the edge lengths of the five solids, inscribed in the same sphere (can you do this?), and an addendum proves that no other regular solids exist (as seen earlier, the proof relies on XI.21).

Other work. Euclid produced a number of other geometric treatises: *Data* (on plane geometry), *On divisions of figures*, *Pseudaria* (a 'book of fallacies', or examples of 'bad geometry', for students; this has been lost), and *Porisms*. This has also been lost, and conjecturally dealt with projective geometry (including constructions of loci; a *porism* is somewhere between an existence theorem and a construction, e.g., find the center of a given circle [Heath p.263].) Four books on *Conics* were later completed by Apollonius. There are also two applied tracts: *Phaenomena* (on spherical geometry for astronomical purposes) and *Optics*.

4- ARCHIMEDES (References: [Dunham 2], [Heath])

Archimedes (287-212 B.C.) was born and lived his whole life in Syracuse (Sicily). His father Phidias was an astronomer, and Archimedes had a life-long interest in astronomy. As a young man, he studied in Alexandria (Egypt). His exploits in applied science in the service of King Hieron of Syracuse, especially in the military realm, made a deep impression on Roman historians (see the account in [Dunham 2, pp. 84-88].) Archimedes was killed during the siege of Syracuse by the Roman general Marcellus (second Punic war, Rome vs. Carthage). Apparently a Roman soldier with orders to take him to Marcellus was told by Archimedes to wait until he had finished a proof he was working on; the soldier got enraged, and killed him on the spot.

Archimedes produced many treatises with deep results in pure and applied mathematics, which would remain unsurpassed for almost 2,000 years; among other things, he developed theoretical mechanics and hydrostatics, as well as an early version of the integral calculus. He is widely (and justly) regarded as the greatest mathematician of Antiquity, and one of the top 4 or 5 of all time.

In the *Method*, Archimedes finds: (1) the area of any segment of a parabola; (2) the volume of a spheroid of revolution; (3) the volume of any segment of a paraboloid or hyperboloid of revolution; (4) the center of gravity of any of the three just-named solids. He includes both rigorous proofs by the method of exhaustion, and 'mechanical proofs', which is how he says these results were found.

In *On the Sphere and the Cylinder*, Archimedes proves: (1) the surface area of a sphere equals four times that of a great circle in it; (2) the surface of a segment of a sphere is equal to a circle with radius equal to the straight line drawn from the vertex of the segment to a point on the circumference of the base; (3) if about a sphere one circumscribes a cylinder with height equal to the diameter of the sphere, then (a) the volume and (b) the surface area of the cylinder are one and one-half times the volume and surface area (respectively) of the sphere.

Archimedes was so fond of this result he left instructions with his friends and relatives that the corresponding figure be carved on his tombstone. Apparently many years later the Roman Cicero found the tomb in a poor state and repaired it, but after that it was lost.

Exercise: verify (3), and use Calculus to verify (2).

In *Measurement of a Circle*, he proves (1) The area of any circle is equal to a right-angled triangle of which one of the sides about the right angle equals the radius, the other equals the circumference of the circle; (2) The ratio of the circumference of a circle to its diameter is less than $3\frac{1}{4}$, but greater than $3\frac{10}{71}$. (That is, between 3.1408... and 3.1428...)

Math: proof of (1) by the method of exhaustion [Dunham 2], discussion of (2).

In *Plane Equilibriums*, Archimedes establishes the fundamental theorems of mechanics, on the basis of simple postulates. Main theorem: Two magnitudes, whether commensurable or incommensurable, balance at distances reciprocally proportional to their magnitudes ('law of the lever'.) This is followed by a computation of centers of gravity for a large number of plane figures.

The quadrature of a parabola contains two solutions to the problem of finding the area of a parabolic segment: by the method of exhaustion, and by a mechanical argument.

Math: argument by the method of exhaustion [Heath pp. 331-332].

In *On Floating Bodies I, II*, Archimedes lays down the foundations of hydrostatics. Proposition 2: *the surface of any fluid at rest is a sphere, the center of which is the same as that of the earth.* Proposition 7 yields a method to solve the problem of King Hieron's crown (the *Eureka!* story.) Book II contains a complete investigation of the conditions of rest and stability of a right segment of a paraboloid of revolution floating in a fluid, for different values of the specific gravity, including the amazing result below.

Math: in some cases, the position of stability does not have the axis of the paraboloid aligned with the vertical (precise statement, [Heath p. 335].)

Other work: *On Spirals, the Cattle Problem, The Sand Reckoner.* Fragments of lost work have survived. For example, Pappus reports Archimedes found 13 'semi-regular' polyhedra: the faces are equilateral and equiangular polygons, but are different ones.

Math: description of the Archimedean polyhedra, and of Kepler's generation method (truncation of the platonic solids), which yields all but two of them [Heath p. 337-338].

A fragment *On the Heptagon in a Circle* survived only in Arabic translation. It contains a method to inscribe a heptagon in a given circle, using conics (hyperbolas).

Math: description of the construction (from [v. der Waerden])

Eratosthenes of Cyrene was a slightly younger contemporary of Archimedes. He is remembered mainly by two accomplishments: (i) his 'sieve' method to list the prime numbers; (ii) measuring the size of the earth. (He found a diameter of 7.850 miles, which is only 50 miles shorter than the true polar diameter [Heath p. 344])

