SOME IMPORTANT PROPOSITIONS IN NEWTON’S *Principia* IN MODERN NOTATION

Ordinarily the derivations below would be included in a course on multivariable calculus, differential equations or basic physics. The fact that we can completely describe the motion of celestial bodies from a fundamental principle (an inverse-square law for gravity) and the tools of Calculus is a fundamental scientific triumph, and one of the main reasons Calculus was quickly accepted as important. Maybe you missed what follows in your education— the exercises will give you an opportunity to close this gap.

1) *Motion under a central force.* Let \( \mathbf{r}(t) \in \mathbb{R}^3, \mathbf{r}'(t) = \mathbf{v}(t), \mathbf{r}''(t) = \mathbf{a}(t) \) be the position, velocity and acceleration vectors. Then the time derivative of the area swept by \( \mathbf{r}(t) \) is:

\[
\frac{dA}{dt} = ||\mathbf{r} \times \mathbf{v}||,
\]

(the norm of the vector product of position and velocity), while the unit normal to the ‘instantaneous plane of motion’ (the plane through the origin containing \( \mathbf{r}(t) \) and \( \mathbf{v}(t) \)) is \( \mathbf{N}(t) \), defined by the equation:

\[
\mathbf{r} \times \mathbf{v} = ||\mathbf{r} \times \mathbf{v}||\mathbf{N}.
\]

Thus we see that the motion takes place in a fixed plane and sweeps out areas at a constant rate exactly when:

\[
\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = 0.
\]

**Exercise 1.** Use Newton’s 2nd. law of motion in the form

\[
\mathbf{F}(\mathbf{r}) = m\mathbf{a}
\]

to show that this happens if, and only if, \( \mathbf{r} \times \mathbf{F}(\mathbf{r}) = 0 \), that is, if \( \mathbf{F} \) is a ‘central force’ (meaning \( \mathbf{F}(\mathbf{r}) \) and \( \mathbf{r} \) are always collinear.)

2) *The equations of motion.* So now we assume the motion is planar, and write down the equations of motion in polar coordinates \((r, \theta)\). Introduce the ‘unit radial’ and ‘unit angular’ vector fields \( \mathbf{u}_r, \mathbf{u}_\theta \):

\[
\mathbf{u}_r = (\cos \theta, \sin \theta), \quad \mathbf{u}_\theta = (-\sin \theta, \cos \theta).
\]
It is easy to see that their time derivatives along the motion are:

\[ u'_r = \theta' u_\theta, \quad u'_\theta = -\theta' u_r. \]

Combined with \( r = ru_r \), this gives the decomposition for the velocity vector:

\[ v = r'u_r + r\theta'u_\theta, \]

while for the acceleration vector:

\[ a = (r'' - r(\theta')^2)u_r + (r\theta'' + 2r'\theta')u_\theta. \]

From the decompositions of \( r \) and \( v \), we see that:

\[ r \times v = r^2 \theta' u_r \times u_\theta = r^2 \theta' N. \]

As seen above in 1) in the case of motion under a central force this vector is constant; in particular, the angular momentum:

\[ L = mr^2 \theta' \]

is a constant of motion.

**Exercise 2.** (i) Show directly (by taking time derivative of \( L \)) that \( L \) is constant in time if and only if the \( u_\theta \) component of \( a \) vanishes (i.e., if and only if \( a \) is purely radial, consistent with \( F \) being a central force and \( F = ma \)). (ii) Show that, in this case, the equation of motion \( F = ma \) can be written in scalar form as:

\[ mr'' = f(r) + \frac{L^2}{mr^3}, \quad \theta' = \frac{L}{mr^2}, \]

where the force is given by \( F(r) = f(r)u_r \).

3) Energy. The speed \( v = ||v|| \) admits the expression:

\[ v^2 = (r')^2 + (r\theta')^2 = (r')^2 + \frac{L^2}{m^2r^2}. \]

Let \( F(r) \) be an antiderivative of \( f(r) \), \( F' = f \). Define the total energy by:

\[ E = \frac{1}{2}mv^2 - F(r) = \frac{1}{2}m(r')^2 + \frac{L^2}{2mr^2} - F(r). \]
Exercise 3. Show (by taking time derivative and using the scalar equations of motion given above) that $E$ is constant throughout the motion ($E'(t) \equiv 0$.)

Define $U(r) = \frac{L^2}{2mr^2} - F(r)$, and let $E_0$ be the constant value of the energy for a given orbit. Then, by definition of the energy:

$$\frac{m(r')^2}{2} = E_0 - U(r),$$

or $r' = \left[\frac{2}{m} (E_0 - U(r))\right]^{1/2}$.

Combined with $\theta' = \frac{L}{mr^2}$, this gives the differential equation for the trajectory:

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{2m r^4}{L^2} (E_0 - U(r)).$$

4) Conics. The general equation of a conic in polar coordinates, with the origin at a focus, is:

$$r(\theta) = \frac{p}{1 + e \cos \theta},$$

where $e$ is the eccentricity ($0 < e < 1$ for an ellipse, $e > 1$ for a hyperbola) and $p$ is the latus rectum (a geometric parameter.)

Exercise 4. Derive from this expression the following:

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{e^2 r^4}{p^2} \left[1 - \frac{1}{e^2} - \frac{p^2}{e^2 r^2} + \frac{2p}{e^2 r}\right].$$

5) Comparing the conclusions of (3) and (4), we find:

$$\frac{2mp^2}{L^2 e^2} (E_0 - U(r)) = 1 - \frac{1}{e^2} - \frac{p^2}{e^2 r^2} + \frac{2p}{e^2 r},$$

where $U(r)$ has the expression given above: $U(r) = \frac{L^2}{2mr^2} - F(r)$. We see that the terms in $1/r^2$ cancel, so we have:

$$\frac{2mp^2}{L^2 e^2} (E_0 + F(r)) = 1 - \frac{1}{e^2} + \frac{2p}{e^2 r}.$$

Assume $F(r) \to 0$ at infinity. Then, comparing the constant terms on both sides of this equality we have:

$$\frac{2mp^2}{L^2} E_0 = e^2 - 1,$$
and comparing the terms depending on $r$ on both sides:

$$F(r) = \frac{L^2}{mp} \frac{1}{r};$$

differentiating this last equation we find:

$$f(r) = -\frac{L^2}{mp} \frac{1}{r^2}.$$

We conclude that the fact the trajectory is a conic implies the force must follow an inverse-square law (and conversely), and moreover that the geometric parameters $e$ and $p$ of the orbit can be computed from the constants of motion $E_0$ (total energy) and $L$ (angular momentum). (Provided we have an explicit expression for the force.)