SOME IMPORTANT PROPOSITIONS IN NEWTON'S *Principia* IN MODERN NOTATION

Ordinarily the derivations below would be included in a course on multivariable calculus, differential equations or basic physics. The fact that we can completely describe the motion of celestial bodies from a fundamental principle (an inverse-square law for gravity) and the tools of Calculus is a fundamental scientific triumph, and one of the main reasons Calculus was quickly accepted as important. Maybe you missed what follows in your education- the exercises will give you an opportunity to close this gap.

1) *Motion under a central force.* Let $\mathbf{r}(t) \in \mathbb{R}^3$, $\mathbf{r}'(t) = \mathbf{v}(t)$,$\mathbf{r}''(t) = \mathbf{a}(t)$ be the position, velocity and acceleration vectors. Then the time derivative of the area swept by $\mathbf{r}$(t) is:

$$\frac{dA}{dt} = ||\mathbf{r} \times \mathbf{v}||,$$

(the norm of the vector product of position and velocity), while the unit normal to the 'instantaneous plane of motion' (the plane through the origin containing $\mathbf{r}(t)$ and $\mathbf{v}$ (t)) is $\mathbf{N}(t)$, defined by the equation:

$$\mathbf{r} \times \mathbf{v} = ||\mathbf{r} \times \mathbf{v}||\mathbf{N}.$$

Thus we see that the motion takes place in a fixed plane and sweeps out areas at a constant rate exactly when:

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = 0.$$

Exercise 1. Use Newton's 2nd. law of motion in the form

$$\mathbf{F}(\mathbf{r}) = m\mathbf{a}$$

to show that this happens if, and only if, $\mathbf{r} \times \mathbf{F}(\mathbf{r}) \equiv 0$, that is, if $\mathbf{F}$ is a 'central force' (meaning $\mathbf{F}(\mathbf{r})$ and $\mathbf{r}$ are always collinear.)

2) *The equations of motion.* So now we assume the motion is planar, and write down the equations of motion in polar coordinates (r, θ). Introduce the 'unit radial' and 'unit angular' vector fields $\mathbf{u}_r$,$\mathbf{u}_\theta$:

$$\mathbf{u}_r = (\cos\theta, \sin\theta), \quad \mathbf{u}_\theta = (-\sin\theta, \cos\theta).$$

It is easy to see that their time derivatives along the motion are:

$$\mathbf{u}_r' = \theta' \mathbf{u}_\theta, \quad \mathbf{u}_\theta' = -\theta' \mathbf{u}_r.$$

Combined with $\mathbf{r} = r\mathbf{u}_r$, this gives the decomposition for the velocity vector:

$$\mathbf{v} = r'\mathbf{u}_r + r\theta'\mathbf{u}_\theta,$$

while for the acceleration vector:

$$\mathbf{a} = (r'' - r(\theta')^2)\mathbf{u}_r + (r\theta'' + 2r'\theta')\mathbf{u}_\theta.$$

From the decompositions of $\mathbf{r}$ and $\mathbf{v}$, we see that:

$$\mathbf{r} \times \mathbf{v} = r^2\theta' \mathbf{u}_r \times \mathbf{u}_\theta = r^2\theta' \mathbf{N}.$$

As seen above in 1) in the case of motion under a central force this vector is constant; in particular, the *angular momentum*:

$$L = mr^2\theta'$$

is a constant of motion.

Exercise 2. (i)Show directly (by taking time derivative of L) that L is constant in time if and only if the $\mathbf{u}_\theta$ component of $\mathbf{a}$ vanishes (i.e., if and only if $\mathbf{a}$ is purely radial, consistent with $\mathbf{F}$ being a central force and $\mathbf{F} = m\mathbf{a}$). (ii) Show that, in this case, the equation of motion $\mathbf{F} = m\mathbf{a}$ can be written in scalar form as:

$$mr'' = f(r) + \frac{L^2}{mr^3}, \quad \theta' = \frac{L}{mr^2},$$

where the force is given by $\mathbf{F}(r) = f(r)\mathbf{u}_r$.

3)*Energy.* The *speed* $v = ||\mathbf{v}||$ admits the expression:

$$v^2 = (r')^2 + (r\theta')^2 = (r')^2 + \frac{L^2}{m^2r^2}.$$

Let $F(r)$ be an antiderivative of $f(r)$, $F' = f$. Define the *total energy* by:

$$E = \frac{1}{2}mv^2 - F(r) = \frac{1}{2}m(r')^2 + \frac{L^2}{2mr^2} - F(r).$$

Exercise 3. Show (by taking time derivative and using the scalar equations of motion given above) that E is constant throughout the motion ($E'(t) \equiv 0$.)

Define $U(r) = \frac{L^2}{2mr^2} - F(r)$, and let E_0 be the constant value of the energy for a given orbit. Then, by definition of the energy:

$$\frac{m(r')^2}{2} = E_0 - U(r), \text{ or } r' = [\frac{2}{m}(E_0 - U(r))]^{1/2}.$$

Combined with $\theta' = \frac{L}{mr^2}$, this gives the differential equation for the trajectory:

$$(\frac{dr}{d\theta})^2 = \frac{2mr^4}{L^2}(E_0 - U(r)).$$

4) *Conics.* The general equation of a conic in polar coordinates, with the origin at a focus, is:

$$r(\theta) = \frac{p}{1 + e\cos\theta},$$

where e is the eccentricity ($0 < e < 1$ for an ellipse, $e > 1$ for a hyperbola) and p is the *latus rectum* (a geometric parameter.)

Exercise 4. Derive from this expression the following:

$$(\frac{dr}{d\theta})^2 = \frac{e^2r^4}{p^2}[1 - \frac{1}{e^2} - \frac{p^2}{e^2r^2} + \frac{2p}{e^2r}].$$

5) Comparing the conclusions of (3) and (4), we find:

$$\frac{2mp^2}{L^2e^2}(E_0 - U(r)) = 1 - \frac{1}{e^2} - \frac{p^2}{e^2r^2} + \frac{2p}{e^2r},$$

where $U(r)$ has the expression given above: $U(r) = L^2/(2mr^2) - F(r)$. We see that the terms in $1/r^2$ cancel, so we have:

$$\frac{2mp^2}{L^2e^2}(E_0 + F(r)) = 1 - \frac{1}{e^2} + \frac{2p}{e^2r}.$$

Assume $F(r) \to 0$ at infinity. Then, comparing the constant terms on both sides of this equality we have:

$$\frac{2mp^2}{L^2}E_0 = e^2 - 1,$$

and comparing the terms depending on r on both sides:

$$F(r) = \frac{L^2}{mp}\frac{1}{r};$$

differentiating this last equation we find:

$$f(r) = -\frac{L^2}{mp}\frac{1}{r^2}.$$

We conclude that the fact the trajectory is a conic implies the force must follow an inverse-square law (and conversely), and moreover that the geometric parameters e and p of the orbit can be computed from the constants of motion E_0 (total energy) and L (angular momentum). (Provided we have an explicit expression for the force.)