

Fourier series: Fourier, Dirichlet, Poisson, Sturm, Liouville

Joseph Fourier (1768-1830) upon returning from Egypt in 1801 was appointed by Napoleon Prefect of the Department of Isères (where Grenoble is the main city). He served in this office for 14 years. Beginning in 1804 he investigated the equation of heat propagation, and submitted in 1807 a memoir to the Academy of Sciences with his results, where it was received with skepticism. Laplace, Lagrange, Poisson and Monge reviewed it. Laplace, for example, could not believe $\cos x$ could be expressed as a sine series, while Laplace and Poisson thought the derivation of the equation lacked in rigor. The memoir was resubmitted successfully in 1811 (for a competition devised for this purpose; Fourier got the prize), and eventually published in 1822.

The main subject is the equation for the temperature $u(x, t)$ in a one-dimensional medium:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$
$$u(x, 0) = f(x) \quad \text{given,}$$

either on the whole real line or on an interval $0 < x < L$ with suitable boundary conditions. The solution is given in terms of Fourier series:

$$u(x, t) = \sum_{n=0}^{\infty} e^{-\left(\frac{4\pi^2 k}{L^2} n^2 t\right)} \left(a_n \cos\left(\frac{2n\pi x}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right) \right)$$

(for periodic boundary conditions), where:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2} \cos\left(\frac{2n\pi x}{L}\right) + \frac{b_n}{2} \sin\left(\frac{2n\pi x}{L}\right),$$

with the expression for the coefficients (where we assume $L = 2\pi$ and f is periodic with period 2π):

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) dx, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx.$$

(Incidentally, the notational idea of attaching the limits of integration to the bottom and top of Leibniz's \int sign first appeared in Fourier's work.)

The question of representability of a continuous function by its Fourier series was considered by Fourier in his treatise, and he believed he had demonstrated it. The key point in the argument was his claim that, for continuous f and 'infinitely large' p one has:

$$\int \sin[p(x - y)] f(y) dy = 0.$$

The idea is that the contribution between two zeros of the sine function is cancelled by the contributions between the next two zeros, since the distance between zeros is infinitely small and the function f (being continuous) is essentially constant over such an interval. This argument is not rigorous, and indeed the result is false: *Du-Bois Raymond* in 1876 gave an example of a continuous 2π -periodic function with the property that a sequence of partial sums $S_n(x)$ at zero of its Fourier series tends to infinity:

$$\lim_{i \rightarrow \infty} |S_{n_i}|(0) = \infty, \quad n_i \rightarrow \infty.$$

The first (and still the most useful) satisfactory result on the representation of periodic functions by Fourier series is due to *P.G.Lejeune Dirichlet* (1805-1859), in 1829:

Theorem: Assume $f(x)$ is 2π -periodic, bounded and continuous with bounded, continuous first derivative except at a finite number of points in $[0, 2\pi]$, where the one-sided limits exist. Then $S_n(x) \rightarrow f(x)$, except where f is discontinuous (where it converges to the average of the one-sided limits.)

Dirichlet had spent the years 1822-1827 in Paris (supporting himself by tutoring), and during this time he had close contact with Fourier's school. Even as a student he must have made an impression, for on his return he received teaching appointments at Breslau (1827) and Berlin (1829), on Alexander v. Humboldt's suggestion. In the years 1847-1849, in Berlin, he taught *Bernhard Riemann* (1826-1866), a 'visiting doctoral student' from Göttingen. In 1855, Dirichlet was appointed to succeed Gauss in Göttingen.

In addition to his fundamental contributions to analysis, Dirichlet also proved a major result in analytic number theory: in a paper of 1840 he solved a fifty-year old conjecture of Legendre: if a and d are co-prime, the arithmetic progression with first term a and differences d contains infinitely many primes; the proof combines a generalization of Euler's product formula with a generalized zeta function, henceforth known as 'Dirichlet L-function'. In 1833, *George Green* (1793-1841) first used what came to be known as the '*Dirichlet principle*' to give a variational solution to a problem in potential theory: for suitable boundary conditions, the variational problem in a domain $D \subset \mathbb{R}^n$:

$$\text{minimize } \int_D |\nabla u|^2 dx$$

has a solution. This 'principle' was used by Dirichlet in potential theory, and later by Riemann for existence proofs in complex function theory; it was

later attacked by an example of Weierstrass (1870), and eventually rescued by Hilbert (1899).

Dirichlet was Riemann's teacher, and Riemann was very interested in the question of *necessary* conditions for representability by Fourier series. Riemann's investigations were published in the 1853 Habilitation paper '*On the representability of a function by means of a trigonometric series*', and led naturally to Riemann's formulation of what we now know as the 'Riemann integral' (see [Dunham-Gallery]), as follows.

The key issue is whether:

$$\lim_{n \rightarrow \infty} \int_0^{2\pi} f(x) \sin(nx) dx = 0.$$

(now known as 'Riemann-Lebesgue lemma'.) Riemann argues we can break up the integral into intervals of length $2\pi/n$, and observe that the sine is positive in the first half of such an interval, negative in the second half; so that if in the k^{th} interval:

$$m_k \leq f(x) \leq M_k \text{ and hence } -M_k \leq -f(x) \leq -m_k,$$

the integral over the two halves is less than $2 \frac{M_k - m_k}{n}$, that is: less than the length of the interval times the 'oscillation' of the function over all intervals. Thus if the sum over all intervals of these products tends to zero as $n \rightarrow \infty$ (the Riemann integrability condition), we have that the desired limit is also zero (that is, $a_n, b_n \rightarrow 0$.) Note that this is a 'rigorization' (in the spirit of Dirichlet, Cauchy and Weierstrass) of Fourier's intuitive idea.

Also in Fourier's work one saw for the first time the '*Fourier transform*':

$$\mathcal{F}[f](s) = \hat{f}(s) = \int_{-\infty}^{\infty} f(t) e^{-ist} dt,$$

(defined, say, for f continuous with $|f|$ integrable over the whole real line.) The main application given by Fourier was to solve the heat equation over the whole real line, with given initial condition $f(x)$. The solution formula is:

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f(x - y) e^{-\frac{y^2}{4kt}} dt;$$

the right-hand side is the 'convolution integral' of f and the 'heat kernel':

$$E_{1/\sqrt{2kt}}(x) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}, \quad E_K(x) = \frac{1}{K\sqrt{2\pi}} e^{-\frac{K^2 x^2}{2}}$$

The solution to this problem had previously been given by Laplace (following work of Poisson), and Fourier showed how it would also follow from his methods.

If f is bounded and uniformly continuous, one indeed has $u(x, t) \rightarrow f(x)$ as $t \rightarrow 0+$, uniformly in x . The use of such ‘uniformity’ conditions in analysis was emphasized by *Karl Weierstrass* (1815-1897), who used Fourier’s solution to prove an important theorem:

Theorem (Weierstrass density theorem (1885)). Any continuous function $f(x)$ on a closed interval $[a, b]$ can be approximated arbitrarily closely by polynomials; that is, given $\epsilon > 0$ one may find a polynomial $p(x)$ so that:

$$\max_{x \in [a, b]} |f(x) - p(x)| < \epsilon.$$

Here is the general idea of the proof: using the standard power series for the exponential, one sees that, for n large, the heat kernel $E_K(x)$ is arbitrarily close to the polynomial (uniformly on a large interval $[-R, R]$):

$$Q_{2n}(x) = (1/k\sqrt{2\pi}) \sum_{r=0}^n (-k^2 x^2/2)^r / r!.$$

Now, as just seen, $f(x)$ is uniformly close (in $[a, b]$) to the solution $u(x, t)$, for t sufficiently close to zero, and this solution is the convolution of (a suitable extension to the real line of) f with $E_{1/\sqrt{2kt}}$. Finally, it is easy to see that the convolution of the extended f with a polynomial is itself a polynomial.

One of the problems Fourier studies in his memoir is the cooling of a sphere of radius R , with heat loss at the boundary proportional to the excess temperature:

$$\frac{\partial u}{\partial r} \Big|_{r=R} = -hu(R, t).$$

This gives rise to the rotationally symmetric heat equation:

$$\frac{1}{k} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r},$$

and by separation of variables, eventually to an ‘eigenvalue problem’:

$$\rho'' + \frac{2}{r}\rho' + \lambda^2\rho = 0,$$

with suitable boundary conditions. It turns out the possible values of λ are solutions of the equation:

$$\frac{R\lambda}{\tan R\lambda} = 1 - Rh.$$

Fourier was not able to show that there are no *complex* solutions λ to this equation, as pointed out by *Siméon-Denis Poisson* (1781-1840), in a paper of 1823 including an alternative derivation of the heat equation, along lines suggested by Laplace (1809). A few years later (1826), Poisson found a proof that the eigenfunctions (of this and similar problems) are orthogonal, and the eigenvalues are real- exactly the same proof presented in courses today. Poisson wrote his own *Théorie analytique de la chaleur* (1835), where he introduced the multidimensional version of these orthogonality results.

The general theory of eigenvalue problems of this type (for second-order ordinary differential equations) was developed by *Joseph Liouville* (1809-1882) and *Jacques-Charles-Francois Sturm* (1803-1855), in a series of papers in the 1830s. A little more precisely, two major memoirs by Sturm (1836) developed completely the oscillation theorems and spectral theory for the whole class of ‘Sturm-Liouville eigenvalue problems’, generalizing the work of Fourier and Poisson. They were followed by three papers by Liouville (in 1836 and 1837), establishing the convergence of the associated generalized Fourier series, and by a joint paper in 1837 (Sturm and Liouville were friends; the latter supported the election of the former, who was Swiss, to the French Academy.)

Poisson made many important contributions to the use of trigonometric series in problems of mathematical physics (and also to analytical mechanics and probability.) For example, the method of ‘*Poisson summation*’ obtains the representation of a 2π - periodic function by its (complex) Fourier series:

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

as the boundary limit of functions defined in the unit disk. Poisson showed that:

$$g(r, x) := \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{inx}$$

converges uniformly in x for each $0 < r < 1$; and:

$$g(r, x) \rightarrow f(x) \text{ as } r \rightarrow 1_-,$$

uniformly in x . In the process, he discovered the famous ‘Poisson kernel’:

$$P_r(x) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx} = \frac{1 - r^2}{1 - 2r \cos x + r^2},$$

which is used to solve the ‘Dirichlet problem’ (find a harmonic function with given boundary values) for the unit disk.

Poisson’s result shows that:

$$\lim_{r \rightarrow 1^-} \lim_{N \rightarrow \infty} \sum_{n=-N}^N c_n r^{|n|} e^{inx}$$

exists, with value $f(x)$; this is not the same as saying that

$$\lim_{N \rightarrow \infty} \lim_{r \rightarrow 1^-} \sum_{n=-N}^N c_n r^{|n|} e^{inx}$$

exists, for arbitrary continuous f ; indeed this is false, by the du Bois Raymond example mentioned earlier. Poisson failed to appreciate this subtle point, and believed he had proved the convergence of all Fourier series (something that also Dirichlet expected to be true.)

In the period between Cauchy and Weierstrass, the idea of rigor in Analysis was in a state of flux, as evidenced by this quote, from a letter of Jacobi to Alexander v. Humboldt, December 1846 (quoted in [Laugwitz, p.63]):

*Only Dirichlet-not I, not Cauchy, not Gauss- knows what a perfectly rigorous proof is; we learn it only from him. When Gauss says he has proved something, I think it very likely; when Cauchy says it, it is a fifty-fifty bet; when Dirichlet says it, it is **certain**. I’d rather not go into these delicate matters...*

SOURCES: 1. T.W.Körner, *Fourier Analysis*, Cambridge U.P.1988; 2. D.Laugwitz, *Bernhard Riemann- Turning Points in the Conception of Mathematics*, Birkhäuser Boston 1999 (Abe Shenitzer, transl.)3. J.Lützen, *Joseph Liouville-Master of Pure and Applied Mathematics*, Springer-Verlag 1990.