Math 241, Spring 2006 - NAME: Freire

List 7: line integrals, conservative vector fields, potentials, Green’s theorem

1. Compute the line integral \( \int_C \mathbf{v} \cdot d\mathbf{r} \), with \( C \) parametrized in two ways: as given, and as a graph over the x-axis.

\[ \mathbf{v} = (\sqrt{y}, x^3 + y), \quad \mathbf{r}(t) = (t^2, t^3), \quad t \in [0, 1], \]
\[ \mathbf{v} = (x^2 - 2xy, x^3 + y), \quad \mathbf{r}(t) = (t^3, t^6), \quad t \in [-1, 1], \]
\[ \mathbf{v} = (x^2 - 2xy, xy - y), \quad \mathbf{r}(t) = (t^2, t^6), \quad t \in [0, 1], \]

2. Find the work done by the force field \( \mathbf{f} \) in moving a particle along the path \( C \) described.

\( \mathbf{f} = (y^2, z^2, x^2); C: \) intersection of unit sphere with cylinder \( x^2 + y^2 = 1 \), counterclockwise viewed from above.

\( \mathbf{f} = (yz, xz, x(y + 1)) \), \( C: \) triangle \((0, 0, 0), (1, 1, 1), (-1, 1, 1) \) (In this order).

\( \mathbf{f} = (x, y, xz - y) \), \( C: \) line segment from the origin to the point \((1, 2, 4)\)

3. Check if the following vector field is a gradient, and, if it isn’t, exhibit a closed curve along which its line integral is not zero.

\( (i) \mathbf{v} = (y^2, y - x) \quad (ii) \mathbf{w} = (xy, x^2, z^2). \)

\( (i) \mathbf{v} = (xy, x^2) \quad (ii) \mathbf{w} = (xz, zy, x^2). \)

\( (i) \mathbf{v} = (x^2, -y^2) \quad (ii) \mathbf{w} = (xz, -y, xz). \)

4. Find a potential function for the following radial vector field:
\[ \mathbf{v} = f(r)\mathbf{u}, \text{in } \mathbb{R}^3 - \{0\}. \]
\[ \mathbf{v} = f(r)\mathbf{u}, \text{in } \mathbb{R}^2 - \{0\}. \]
\[ \mathbf{v} = c\frac{r}{r^3} \text{in } \mathbb{R}^3 - \{0\}. \]

5. Determine whether each of the following vector fields is a gradient; if it is, find a potential function for it.

\[ \mathbf{v} = (3x^2y, x^3); \quad \mathbf{w} = (2x^2 + 8xy^2, 3x^3y - 3xy, -4y^2z^2 - 2x^3z). \]
\[ \mathbf{v} = (2xe^y + y, x^2y + x - 2y); \quad \mathbf{w} = (y^2 \cos x + z^3, -4 + 2y \sin x, 3xz^2 + 2). \]
\[ \mathbf{v} = (\sin y - y \sin x + x, \cos x + x \cos y + y); \quad \mathbf{w} = (4xy - 3x^2z^2, 2x^2, -2x^3z - 3z^2). \]

6. Let \( \mathbf{v} = (-y/r^2, x/r^2) \) in \( \mathbb{R}^2 - \{0\} \). Compute the line integral of \( \mathbf{v} \) along the following curves (justify your answer):
   (i) the unit circle (center at the origin), traversed twice, counterclockwise;
   (ii) the ellipse \( x^2 + 2y^2 = 1 \), traversed three times, clockwise;
   (iii) the circle \( (x - 2)^2 + y^2 = 1 \) traversed one, counterclockwise.

7. Use Green’s theorem to compute the following:
   Work done by the force field \( \mathbf{f}(x, y) = (y + 3x, 2y - x) \) in moving a particle once around the ellipse \( 4x^2 + y^2 = 4 \) (counterclockwise).
   The line integral of \( \mathbf{v} = (y^2, x) \) around the square with vertices \( (\pm 1, \pm 1) \) (counterclockwise):
   The area enclosed by the curve with parametric equations: \( (2 \cos^3 t, 2 \sin^3 t), t \in [0, 2\pi] \).

8. Use Green’s theorem to evaluate:
   The area of a pentagon with vertices \( (0,0), (2,1), (1,3), (0,2), (-1,1) \).
   The centroid of the triangle with vertices \( (0,0), (1,0), (0,1) \).
   The centroid of a semicircular region of radius \( R \).

9. Find the divergence of the following vector field in the plane, and use it to compute its flux across the given curve (with respect to the outward unit normal).
\[ \mathbf{v}(x, y) = (2x, 3y), \quad C : x^2 + 4y^2 = 4. \]

\[ \mathbf{v}(x, y) = (y^2, 2xy), \quad C : x^2 + y^2 = 1. \]

\[ \mathbf{v}(x, y) = (xy, -(1/2)y^2), \quad C : x^4 + y^4 = 1. \]

10. The following vector fields in the plane are either conservative or incompressible (divergence-free). Decide which one, and sketch its flow lines.

\[ \mathbf{v} = (-2x, 2y) \]

\[ \mathbf{v} = (3x, 3y). \]

\[ \mathbf{v} = (-y, x). \]