Math 241, Spring 2006 -NAME: Freire

List 7: line integrals, conservative vector fields, potentials, Green's theorem

1. Compute the line integral $\int_C \mathbf{v} \cdot d\mathbf{r}$, with C parametrized in two ways: as given, and as a graph over the x-axis.

$$\mathbf{v} = (\sqrt{y}, x^3 + y), \quad \mathbf{r}(t) = (t^2, t^3), t \in [0, 1],$$

$$\mathbf{v} = (x^2 - 2xy, x^3 + y), \quad \mathbf{r}(t) = (t^3, t^6), t \in [-1, 1],$$

$$\mathbf{v} = (x^2 - 2xy, xy - y), \quad \mathbf{r}(t) = (t^2, t^6), t \in [0, 1],$$

2. Find the work done by the force field $\mathbf{f}$ in moving a particle along the path C described.

$\mathbf{f} = (y^2, z^2, x^2)$; C: intersection of unit sphere with cylinder $x^2 + y^2 = 1$,

counterclockwise viewed from above.

$\mathbf{f} = (yz, xz, x(y + 1))$, C: triangle $(0, 0, 0), (1, 1, 1), (-1, 1, 1)$ (In this order).

$\mathbf{f} = (x, y, xz - y)$, C: line segment from the origin to the point $(1, 2, 4)$

3. Check if the following vector field is a gradient, and, if it isn't, exhibit a closed curve along which its line integral is not zero.

$$(i)\mathbf{v} = (y^2, y - x) \quad (ii)\mathbf{w} = (xy, x^2, z^2).$$

$$(i)\mathbf{v} = (xy, x^2), \quad (ii)\mathbf{w} = (xz, zy, x^2).$$

$$(i)\mathbf{v} = (x^2, -y^2), \quad (ii)\mathbf{w} = (xz, -y, xz).$$

4. Find a potential function for the following radial vector field:

$$\mathbf{v} = f(r)\mathbf{u}_r \text{in } \mathbb{R}^3 - \{0\}.$$

$$\mathbf{v} = f(r)\mathbf{u}_r \text{in } \mathbb{R}^2 - \{0\}.$$

$$\mathbf{v} = c\frac{\mathbf{r}}{r^3} \text{in } \mathbb{R}^3 - \{0\}.$$

5. Determine whether each of the following vector fields is a gradient; if it is, find a potential function for it.

$$\mathbf{v} = (3x^2y, x^3); \quad \mathbf{w}=(2x^2 + 8xy^2, 3x^3y - 3xy, -4y^2z^2 - 2x^3z).$$

$$\mathbf{v} = (2xe^y + y, x^2y + x - 2y); \quad \mathbf{w} = (y^2\cos x + z^3, -4 + 2y\sin x, 3xz^2 + 2).$$

$$\mathbf{v}=((\sin y - y\sin x + x, \cos x + x\cos y + y), \quad \mathbf{w} = (4xy - 3x^2z^2, 2x^2, -2x^3z - 3z^2).$$

6. Let $\mathbf{v} = (-y/r^2, x/r^2)$ in $\mathbb{R}^2 - \{0\}$. Compute the line integral of $\mathbf{v}$ along the following curves (justify your answer):

(i) the unit circle (center at the origin), traversed twice, counterclockwise;

(ii) the ellipse $x^2 + 2y^2 = 1$, traversed three times, clockwise;

(iii) the circle $(x-2)^2 + y^2 = 1$. traversed one, counterclockwise.

7. Use Green's theorem to compute the following:

Work done by the force field $\mathbf{f}(x, y) = (y+3x, 2y-x)$ in moving a particle once around the ellipse $4x^2 + y^2 = 4$ (counterclockwise).

The line integral of $\mathbf{v} = (y^2, x)$ around the square with vertices $(\pm 1, \pm 1)$ (counterclockwise);

The area enclosed by the curve with parametric equations: $(2\cos^3 t, 2\sin^3 t), t \in [0, 2\pi]$.

8. Use Green's theorem to evaluate:

The area of a pentagon with vertices (0,0), (2,1), (1,3), (0,2), (-1,1).

The centroid of the triangle with vertices (0,0), (1,0), (0,1).

The centroid of a semicircular region of radius R.

9. Find the divergence of the following vector field in the plane, and use it to compute its flux across the given curve (with respect to the outward unit normal).

$$\mathbf{v}(x,y) = (2x, 3y), C : x^2 + 4y^2 = 4.$$

$$\mathbf{v}(x,y) = (y^2, 2xy), C : x^2 + y^2 = 1.$$

$$\mathbf{v}(x,y) = (xy, -(1/2)y^2), C : x^4 + y^4 = 1.$$

10. The following vector fields in the plane are either conservative or incompressible (divergence-free). Decide which one, and sketch its flow lines.

$$\mathbf{v} = (-2x, 2y)$$

$$\mathbf{v} = (3x, 3y).$$

$$\mathbf{v} = (-y, x).$$