LIST 6- multiple integrals: change of variable, improper integrals/ implicit functions/ vector fields: chain rule

1. (change of variable, linear) $S$ is the parallelogram with vertices $0$ $(\pi, 0), (2\pi, \pi), (\pi, 2\pi), (0, \pi)$. Use a linear change of variables to compute:

$$\int \int_{S} (x - y)^2 \sin^2(x + y)dxdy.$$

1. (change of variable, linear) $S$ is the parallelogram with vertices $0, (0, 0), (2, 10), (3, 17), (1, 7)$. Use a linear change of variables to compute:

$$\int \int_{S} xydxdy.$$

(Map $S$ to a rectangle with sides parallel to the axes.)

1. (change of variable, linear) $S$ is the parallelogram with vertices $0, (0, 0), (1, 0), (2, 4), (3, 4)$. Use a linear change of variables to compute:

$$\int \int_{S} (x - \frac{y}{2})dxdy.$$

(Map $S$ to a rectangle with sides parallel to the axes.)

2. (change of variable, nonlinear) Consider the mapping defined by

$$x = u + v, \quad y = v - u^2.$$ 

Let $T$ be the triangle in the $(u, v)$ plane with vertices $(0, 0), (2, 0), (0, 2)$. Sketch the image $S$ of $T$ in the $(x, y)$ plane, and use the mapping to compute the integral:

$$\int \int_{S} xydxdy.$$

2. (change of variable, nonlinear) Consider the mapping defined by:

$$x = u^2 - v^2, \quad y = uv.$$ 

Let $T$ be the unit square $[0, 1] \times [0, 1]$ in the $(u, v)$ plane. Sketch the image $S$ of $T$ in the $(x, y)$ plane, and use the mapping to compute the integral:

$$\int \int_{S} (x + y)dxdy.$$

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2. (change of variable, nonlinear) Consider the mapping defined by:
\[ x = v^2 - 2u^2, \quad y = uv. \]
Let \( T \) be the unit square \([0, 1] \times [0, 1]\) in the \((u, v)\) plane. Sketch the image \( S \) of \( T \) in the \((x, y)\) plane, and use the mapping to compute the integral:
\[ \int \int_S (x + y) \, dx \, dy. \]

3. (change of variables in triple integrals) (i) Compute the centroid of the region:
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1, \quad z \geq 0. \]
(ii) Compute \( \int \int \int x^2 + yz \, dx \, dy \, dz \) over the same region. (Remark: symmetry.)

3. (change of variables in triple integrals) Compute the triple integral:
\[ \int \int \int_T |xyz| \, dx \, dy \, dz, \]
where \( T \) is the solid ellipsoid \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1. \)

3. (change of variables in triple integrals) Compute the triple integral:
\[ \int \int \int_P xyz \, dx \, dy \, dz, \]
where \( P \) is the solid ellipsoid in the first octant: \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1, x \geq 0, y \geq 0, z \geq 0. \)

4. (integration in spherical coordinates) Compute in spherical coordinates: (i) \( \int \int_B x^2 + y^2 + z^2 \, dV \); \( B \) is the unit ball; (ii) \( \int \int_T e^{\sqrt{x^2 + y^2 + z^2}} \, dV \); \( T \) is the part of the unit ball in the first octant.

4. (integration in spherical coordinates) Compute in spherical coordinates: (i) \( \int \int_T zdV \); (ii) the centroid of \( T \).
\[ T = \{(x, y, z) | 1 \leq x^2 + y^2 + z^2 \leq 4, z \geq 0 \}. \]

4. (integration in spherical coordinates) Compute in spherical coordinates: (i) \( \int \int_T zdV \); (ii) the centroid of \( T \).
\[ T = \{(x, y, z) | 1 \leq x^2 + y^2 + z^2 \leq 9, x \geq 0, y \geq 0, z \geq 0 \}. \]
5. (improper multiple integrals) Decide whether the following integrals are convergent or divergent (there is no need to compute its value, if convergent).

(i) \( \int \int_{R^2} \frac{1}{(1 + x^2 + y^2)^{5/2}} \, dA \);

(ii) \( \int \int_{R^2} \frac{1}{1 + x^2 + y^2} \, dA \);

(iii) \( \int \int \int_{R^3} \frac{1}{(1 + x^2 + y^2 + z^2)^{3/2}} \, dV \).

5. (improper multiple integrals) Decide whether the following integrals are convergent or divergent (there is no need to compute its value, if convergent).

(i) \( \int \int_{R^2} \frac{1}{(2 + x^2 + y^2)^{3/2}} \, dA \);

(ii) \( \int \int_{R^2} \frac{1}{(3 + x^2 + y^2)^{1/2}} \, dA \);

(iii) \( \int \int \int_{R^3} \frac{1}{(1 + x^2 + y^2 + z^2)^2} \, dV \).

5. (improper multiple integrals) Decide whether the following integrals are convergent or divergent (there is no need to compute its value, if convergent).

(i) \( \int \int_{R^2} \frac{1}{(3 + x^2 + y^2)^{1/2}} \, dA \);

(ii) \( \int \int \int_{R^3} \frac{1}{(1 + x^2 + y^2 + z^2)^{3/2}} \, dV \);

(iii) \( \int \int \int_{R^3} \frac{1}{(10 + x^2 + y^2 + z^2)^9} \, dV \).

6. (functions defined implicitly) (i) The equations \( 2x = u^2 - v^2, y = uv \) define \( u \) and \( v \) as functions of \( x, y \). Find formulas for the first partial derivatives \( u_x, u_y, v_x, v_y \).

(ii) Given a function of two variables \( F(x_1, x_2) \), define a function \( u(x, y) \) implicitly by the relation:

\[ u = F(x + u, yu) \]

Find \( u_x \) and \( u_y \) in terms of the partial derivatives of \( F \).
6. 

(functions defined implicitly) The equations \( x + y = uv, xy = u - v \) determine \( x \) and \( y \) implicitly as functions of \( u, v \). Show that \( x_u = (xv - 1)/(x - y) \) for \( x \neq y \), and find similar formulas for \( x_v, y_u, y_v \).

6. 

(functions defined implicitly) The equations \( x - y = uv, xy = u + v \) determine \( x \) and \( y \) implicitly as functions of \( u, v \). Show that \( x_u = (xv + 1)/(x + y) \) for \( x \neq -y \), and find similar formulas for \( x_v, y_u, y_v \).

7.

curves defined as an intersection. The surfaces \( 2x^2 + 3y^2 - z^2 = 25 \) and \( x^2 + y^2 = z^2 \) intersect along a curve \( C \) which passes through the point \( P(\sqrt{3}, 3, 4) \). Find a unit tangent vector \( T \) to \( C \) at \( P \).

7.

curves defined as an intersection. The surfaces \( x^2 + 3y^2 - z^2 = 4 \) and \( x^2 + y^2 = z^2 \) intersect along a curve \( C \) which passes through the point \( P(\sqrt{2}, \sqrt{2}, 2) \). Find a unit tangent vector \( T \) to \( C \) at \( P \).

7.

curves defined as an intersection. The surfaces \( 23x^2 + 2y^2 - z^2 = 25 \) and \( x^2 + y^2 = z^2 \) intersect along a curve \( C \) which passes through the point \( P(3, \sqrt{7}, 4) \). Find a unit tangent vector \( T \) to \( C \) at \( P \).

8. 

(functions defined implicitly.) The equation \( x + z + (y + z)^2 = 6 \) defines \( z \) implicitly as a function of \( x, y \), say \( z = f(x, y) \). Compute \( f_x, f_y \) and \( f_{xy} \) in terms of \( x, y \) and \( z \).

8. 

(functions defined implicitly.) The equation \( (x + z)^2 + y + z = 8 \) defines \( z \) implicitly as a function of \( x, y \), say \( z = f(x, y) \). Compute \( f_x, f_y \) and \( f_{xy} \) in terms of \( x, y \) and \( z \).

8. 

(functions defined implicitly.) The equation \( (x - z)^2 + y - z = 1 \) defines \( z \) implicitly as a function of \( x, y \), say \( z = f(x, y) \). Compute \( f_x, f_y \) and \( f_{xy} \) in terms of \( x, y \) and \( z \).

9. (Chain rule for maps I) The equations below define a map \( (s, t) \mapsto (x, y) \). If \( u = f(x, y) \), the composition defines a function \( u(s, t) \). Find the first partial derivatives of \( u \) in terms of those of \( f \).

\[
x(s, t) = s + t, \quad y(s, t) = st.
\]

9. (Chain rule for maps I) The equations below define a map \( (s, t) \mapsto (x, y) \). If \( u = f(x, y) \), the composition defines a function \( u(s, t) \). Find the first partial derivatives of \( u \) in terms of those of \( f \).

\[
x(s, t) = st, \quad y(s, t) = s/t.
\]

9. (Chain rule for maps I) The equations below define a map \( (s, t) \mapsto (x, y) \). If \( u = f(x, y) \), the composition defines a function \( u(s, t) \). Find the first partial derivatives of \( u \) in terms of those of \( f \).

\[
x(s, t) = s - t, \quad y(s, t) = s + t.
\]
10. (Chain rule for maps II) The equations below define a map \((r, s, t) \mapsto (x, y, z)\). If \(u = f(x, y, z)\), the composition defines a function \(u(r, s, t)\). Find the first partial derivatives of \(u\) in terms of those of \(f\).

\[
\begin{align*}
x(r, s, t) &= r + s + t, & y(r, s, t) &= r - 2s + 3t, & z(r, s, t) &= 2r + s - t \\
x(r, s, t) &= 2r + s + t, & y(r, s, t) &= r - s + t, & z(r, s, t) &= 2r + 3s - t \\
x(r, s, t) &= r^2 + s^2 + t^2, & y(r, s, t) &= r^2 - s^2 + t^2, & z(r, s, t) &= r^2 - s^2 - t^2 \\
x(s, t) &= s^2 + t^2, & y(s, t) &= s^2 - t^2, & z(s, t) &= 2st \\
x(s, t) &= s + t, & y(s, t) &= s - t, & 2st \\
x(s, t) &= t^2 - s^2, & y(s, t) &= t^2 + s^2, & z(s, t) &= st
\end{align*}
\]