MATH 231- Spring 2008- second test, 4/3/2008.

1. Consider the system $X' = \begin{bmatrix} 2 & -4 \\ 5 & -2 \end{bmatrix}$X, $X = X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \in \mathbb{R}^2$

(i) [10] Find the general solution ***in real form*** (the eigenvalues are $\pm 4i$).

(ii) [8] Check whether $E(x, y)$ is a conserved energy for the system.

$$E(x, y) = 5x^2 - 4xy + 4y^2$$

(iii) [7] Consider the solution with initial condition $x(0) = 1, y(0) = 0$. What is the equation of the corresponding (x, y) curve? Identify the curve and the direction in which it is traversed (for increasing t.)

2. Let $L[y] = y'' - 3y' - 10y$, for $y = y(t)$.

(i) [10] Use Laplace transforms to solve the initial-value problem:

$$L[y] = 0, \quad y(0) = 0, y'(0) = 1.$$

(ii)[10] Solve the initial-value problem:

$$L[y] = f(t), \quad y(0) = y'(0) = 0, \text{ where:}$$

$$f(t) = \begin{cases} 2, t \in [2, 4) \\ 1, t \in [0, 2) \cup [4, \infty) \end{cases}$$

Compute the numerical value of the solution at $t = 3$ and at $t = 5$.

3. Consider the non-linear autonomous equation (of 'Bernouilli type'):

$$y' = 16y - y^5, \quad y = y(t)$$

(i) [10] Find and classify the equilibria (as 'stable' or 'unstable').

(ii)[10] Sketch the diagram of all solutions (in the (t, y) plane), with one solution in each of the regions defined by the equilibria. Indicate in your diagram which solutions are defined for all $t \in \mathbb{R}$, which only for an interval.

(iii)[7] Find an explicit expression for the solution with initial condition $y(0) = 4$, including the interval where it is defined.

4. Let $A = \begin{bmatrix} -3 & -1 \\ -1 & -3 \end{bmatrix}$. The eigenvalues of A are -4 and -2.

(i)[10] Find the general solution of the system $X' = AX$, $X(t) \in \mathbb{R}^2$.

(ii)[10] Sketch the diagram of all solutions in the (x, y) plane; include the eigenspaces in your sketch, and arrows indicating the behavior of solutions for $t > 0$. (Include at least four solutions not contained in the eigenspaces.)

(iii)[8] For the non-homogeneous problem $X' = AX + \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}$, use substitution to derive the second order equation satisfied by the ***second*** component $y(t)$.Are all solutions of this non-homogeneous system ***bounded*** for $t > 0$?