MATH 231- Spring 2008- final exam, 5/1/2008. Closed books, closed notes. Calculators OK. Time given: 120 min (10:15-12:55)

1.[16] A particle executes simple harmonic motion on a line (without friction). At $t = 0$, it is $5m$ from the equilibrium position, its velocity is $8m/s$ and the acceleration is $-20m/s^2$. Find the equation $y(t)$ for position as a function of time.

2.[16] Find **one** solution of the equation:

$$y'' + 2y' + 10y = \sin t, y = y(t).$$

(You may use complex numbers to help, but the answer must be a real-valued function.)

3.[18] Let $A = \begin{bmatrix} -3 & -1 \\ -1 & -3 \end{bmatrix}$. The eigenvalues of A are -4 and -2.

(i)[9] Find the general solution of the system $X' = AX$, $X(t) \in \mathbb{R}^2$.

(ii)[9] Sketch the diagram of all solutions in the (x, y) plane; include the eigenspaces in your sketch, and arrows indicating the behavior of solutions for $t > 0$. (Include at least four solutions not contained in the eigenspaces.)

4.[18] Consider the non-linear autonomous equation (of 'Bernouilli type'):

$$y' = 9y - y^3, \quad y = y(t)$$

(i)[9] Find and classify the equilibria (as 'stable' or 'unstable'). Sketch the diagram of all solutions (in the (t, y) plane), with one solution in each of the regions defined by the equilibria. Indicate in your diagram which solutions are defined for all $t \in \mathbb{R}$, which only for an interval.

(ii)[9] Find an explicit expression for the solution with initial condition $y(0) = 4$, including the interval where it is defined.

5.[16] For the following second-order homogeneous equation ($y = y(t)$) given below, one solution is given. Use the 'reduction of order' method to find a second, linearly independent solution.

$$2t^2y'' + 3ty' - y = 0, \quad y_1 = \frac{1}{t} \quad (t > 0)$$

6.[16] Solve the following initial-value problem, using the fact that the equation given is exact. If possible, find an explicit form $y(t)$ for the solution.

$$3t^2 + 4ty + (2y + 2t^2)y' = 0, \quad y(0) = 1$$