Second-order equations and mechanics.

The general second-order equation:

$$y'' = f(t, y, y'), \quad y = y(t),$$

may be interpreted (via Newton's Second Law) as describing the position $y(t)$ on a line of a particle of unit mass subject to a force f depending on time, position and and velocity. An important special case is:

$$y'' = f(y),$$

where the force depends only on position. In general, mechanical systems without 'friction' or 'energy dissipation' effects follow a law of this type. This is 'conservative dynamics', for there is always a 'conserved quantity' for such motions, that is, a function of two variables $E(y, v)$ such that:

$$\frac{d}{dt} E(y(t), y'(t)) \equiv 0,$$

whenever $y(t)$ is a solution. To find $E(y, v)$ given $f(y)$, let $U(y)$ be an antiderivative of $-f$:

$$U(y) = -\int f(y) dy,$$

and then define E as the sum of two terms:

$$E(y, v) = \frac{1}{2} v^2 + U(y).$$

It is then easy to check that this $E(y, v)$ is a conserved quantity, in the sense defined above (*done in class.*) In Physics terminology, E is the total energy of the solution, $(1/2)(y')^2$ the kinetic energy at time t (recall we assume the mass is 1) and $U(y(t))$ the potential energy at time t. (Although the sum is constant, each of KE and PE depends on t).

In many cases where the equation $y'' = f(y)$ is non-linear (hence not explicitly solvable), it is still possible to say a lot about the solutions just by examining the graph of $U(y)$, if we know the initial energy $E_0 = (1/2)(y'(0))^2 + U(y(0))$. We have the following three general properties (*figure drawn in class: two-well potential).*

1. The critical points of U (equivalently: the zeros of f) are *equilibria* of the system, that is, define constant solutions $y(t) \equiv \bar{y}$ (with initial conditions $y(0) = \bar{y}, y'(0) = 0$).

2. Call an equilibrium $\bar{y}$ *stable* if *any* solution starting close to $\bar{y}$ with small velocity remains close to $\bar{y}$ for all time; *unstable* if *some* solutions starting close to $\bar{y}$ move away from $\bar{y}$ for positive time. Then if $\bar{y}$ is a *local minimum* of U, $\bar{y}$ is stable, if a *local max* of U, $\bar{y}$ is unstable.

By the second derivative test, critical points of U where $d^2U/dy^2 > 0$ are stable, while those where $d^2U/dy^2 < 0$ are unstable. (If $d^2U/dy^2 = 0$ at $\bar{y}$, we can still decide stability based on the graph of U). This is easy to remember if we think of the graph of U as the profile of a 'mountain' (better: a 'smooth track') on which a ball is released at a given point (from rest, if $y'(0) = 0$, or with a 'kick' if $y'(0) \neq 0$) and allowed to roll (assuming no friction.) (*Figure drawn in class.*)

3. Suppose the potential $U(y)$ (assumed to be a smooth function defined on all of $\mathbb{R}$) has the following property:

$$U(y) \to +\infty \text{ as } y \to \pm\infty.$$

(Functions with this property are called 'proper', example: any polynomial of even degree with positive highest-degree coefficient.) Then all solutions are defined for all $t \in \mathbb{R}$ and are *bounded* between two extreme values: $y_{min} \leq y(t) \leq y_{max}$, where y_{min} and y_{max} depend on the solution and can be computed from the initial conditions and U.

This follows from conservation of energy: since the kinetic energy is non-negative, we have:

$$U(y(t)) \leq E(y(t), y'(t)) = E(y(0), y'(0)) = E_0.$$

Since U is proper, for any E_0 the set of y such that $U(y) \leq E_0$ is bounded (or empty), in fact contained in an interval $[y_{min}, y_{max}]$, where $U(y_{min}) = U(y_{max}) = E_0$ (these are points where the particle 'reverses direction', so the velocity is zero).

Remark. If the potential is not 'proper', by examining the graph of U, the value of E_0 and the initial conditions we may still conclude a particular solution is bounded. For example, the potentials $U(y) = y(y^2 - 1)$ and $U(y) = \sin y$ are not proper, but it is easy to identify ICs leading both to bounded and to unbounded solutions (*done in class*). (The 'rolling ball' analogy helps).

There is one more thing to mention: the graphical analysis of conservative systems usually includes drawing a 'phase plane diagram', that is, a

plot of the pair $(y(t), y'(t))$ as a curve in the (y, v) plane. Two curves corresponding to different solutions never intersect (by uniqueness of solutions with given IC), and each solution is contained in a *level set* of the function of two variables $E(y, v)$ (since E is constant throughout the motion.). Thus the level sets $E(y, v) = const$ may be thought of as 'implicit solutions'. The time parameter t is invisible in this diagram, but we can still draw arrows to indicate the direction of evolution of the pair (y, y') as time increases (note y increases when $y' > 0$, decreases when $y' < 0$). Curves (=level sets of $E(y, v)$) corresponding to solutions near a *stable* equilibrium will be closed curves. If the potential is proper (and the number of equilibria is finite), curves representing solutions with sufficiently high total energy will also be closed. In both cases, this corresponds to the fact that the motion is *periodic*. (*Example drawn in class.)*

Example 1. Consider the initial-value problem:

$$y'' = 1 - 3y^2, \quad y = y(t), \quad y(0) = y'(0) = 0.$$

Is the solution bounded? If so, what is its range?

Solution. $U(y) = y^3 - y$ is not proper, but looking at its graph we see that a solution with $E_0 = 0$ is bounded. Indeed since

$$E(y, y') = (1/2)(y')^2 + y^3 - y \leq E_0 = 0,$$

we must have $y(t)^3 - y(t) \leq 0$, so $y(t) \in [0, 1]$. This is the range of the solution.

Example 2. Consider the equation:

$$y'' = 4y(1 - y^2).$$

(1) Find and classify the equilibria. (2) Are all solutions bounded? (3) If $y(0) = 0, y'(0) = 1$, find the range of the solution. (4) Sketch the phase diagram of solutions.

Solution. (1) The equilibria are the zeros of f: $\bar{y} = -1, 0, 1$. With $U(y) = y^4 - 2y^2$, $\bar{y} = \pm 1$ are local minima (hence stable equilibria), while 0 is a local max (hence unstable). (2) Since $U(y) \to +\infty$ as $y \to \pm\infty$, all solutions are bounded. (3) $E(0) = (1/2)(y'(0))^2 + U(y(0)) = 1/2$, so $U(y(t)) \leq 1/2$. To find the set of y where $U(y) \leq 1/2$, we solve the equation (quadratic in y^2):

$$y^4 - 2y^2 - \frac{1}{2} = 0,$$

finding $y^2 = (1+\sqrt{6})/2$ (only the positive solution matters). Hence the range of this solution is the symmetric interval $[-y_m, y_m]$,$y_m = [(1+\sqrt{6})/2]^{1/2}$.
(4) *Sketch shown in class.*