1. THE EXPONENTIAL FUNCTION.

The solution of first-order linear differential equations is based on the exponential, so it is useful to recall its definition and properties. (As a side benefit, we’ll review some important facts from one-variable Calculus).

The exponential function is defined as the inverse of natural logarithm. In turn, the natural logarithm is defined as a definite integral:

\[ L(x) = \int_{1}^{x} \frac{dt}{t}, \quad x > 0. \]

\(L(x)\) is continuous for \(x > 0\) (in fact, differentiable with \(L'(x) = 1/x\), by the fundamental theorem of calculus), positive for \(x > 1\), negative for \(0 < x < 1\), strictly increasing. From this definition we can prove the basic property:

\[ L(xy) = L(x) + L(y) \quad \text{for } x, y > 0. \]

(proof seen in class.)

Recall the inverse function theorem of Calculus:

**Theorem.** (a) If \(f(x)\) is strictly increasing and continuous on an interval \((a, b)\), onto an interval \((c, d)\), then the inverse function \(g(y)\) is defined, strictly increasing and continuous on \((c, d)\). By definition of ‘inverse function’:

\[ g(f(x)) = x \quad \text{for all } x \in (a, b); \quad f(g(y)) = y \quad \text{for all } y \in (c, d). \]

(b) If, in addition, \(f\) is differentiable with \(f'(x) \neq 0\) on \((a, b)\), then \(g\) is differentiable on \((c, d)\) with \(g'(y) = \frac{1}{f'(g(y))}\).

Note that the domain of the inverse function is the interval \((c, d)\) covered by all values of \(f\) on \((a, b)\). What is the interval covered by \(L(x)\) on \([1, \infty)\)? If it were a bounded interval \([0, A]\), this would imply the total area under the graph of \(1/t\) for \(0 < t < \infty\) would be finite (equal to \(A\), which we know is not true (use the integral test and divergence of the harmonic series). Similarly \(L(x)\) for \(x\) in the interval \((0, 1)\) must cover the whole negative real line; otherwise the integral from 0 to 1 of \(1/t\) would be finite, which is not true. We conclude \(L(x)\) is one-to-one onto (‘bijective’) from \((0, \infty)\) to the whole real line. Thus the domain of its inverse is all of \(\mathbb{R}\).

**Definition.** The exponential function \(E : \mathbb{R} \to \mathbb{R}_+\) is the inverse function of the natural logarithm \(L(x)\).
Now, given two real numbers $y_1, y_2$, there are unique positive real numbers $x_1, x_2$ so that $y_1 = L(x_1)$ and $y_2 = L(x_2)$. Then:

$$E(y_1 + y_2) = E(L(x_1) + L(x_2)) = E(L(x_1x_2)) = x_1x_2 = E(y_1)E(y_2),$$

using the basic property of $L(x)$ and the definition of ‘inverse function’. That is, $E$ transforms sums into products. In particular, for any natural number $n$, we find easily that:

$$E(n) = E(1)^n,$$

and similarly for fractional powers $p/q$:

$$E(p/q) = E(1)^{p/q}$$

(since $[E(p/q)]^q = E(p) = E(1)^p$.) Thus, at least for rational numbers $x$, the function $E(x)$ coincides with $A^x$, where $A = E(1)$. Question: what is the number $A$?

To answer this, we have to look at values of $L(x)$ for $x$ close to 1. We start by making a change of variable (for $x > 1$):

$$L(x) = \int_0^{x-1} \frac{dt}{1 + t}.$$

Then recall the expansion, convergent for $|t| < 1$ (geometric series):

$$\frac{1}{1 + t} = 1 - t + t^2 - t^3 + \ldots$$

Integrating:

$$L(1 + a) = \int_0^a (1 - t + t^2 + \ldots)dt = a - \frac{a^2}{2} + \frac{a^3}{3} + \ldots$$

(convergent for $0 < a < 1$). Now use this for $a = 1/n$, $n > 1$ (and multiply by $n$):

$$nL(1 + \frac{1}{n}) = 1 - \frac{1}{2n} + \frac{1}{3n^2} + \ldots$$

We see that:

$$\lim_{n \to \infty} nL(1 + \frac{1}{n}) = 1,$$

or:

$$L[(1 + \frac{1}{n})^n] \to 1 \text{ as } n \to \infty.$$
Now apply the exponential function \( E \) to both sides of this limit (this is legitimate, since \( E \) is a continuous function.). We obtain:

\[
(1 + \frac{1}{n})^n \to E(1) \text{ as } n \to \infty.
\]

Note: this 18th. century argument (L. Euler) is not completely rigorous by late 20th. century standards. By the end of the 21st., who knows?

With this formula we can compute \( E(1) \) to an arbitrary degree of precision, and find it equals the number usually denoted by \( e \):

\[
E(1) = 2.718281828... = e.
\]

This justifies the conventional notation for the exponential function:

\[
E(x) = e^x.
\]

(Keep in mind this is just a notation—giving something a different name—and contains no new mathematical information beyond the definition given above.)

Now consider derivatives. From (ii) in the inverse function theorem, we find, for any \( y \in \mathbb{R} \):

\[
E'(y) = \frac{1}{L'(E(y))} = \frac{1}{1/E(y)} = E(y).
\]

That is, the exponential function equals its own derivative! Another way of saying this is that \( E \) is a solution of the ‘first order differential equation’:

\[
f' = f.
\]

More generally, for any real number \( \alpha \), we have from the chain rule:

\[
E'(\alpha x) = \alpha \frac{d}{dy}E(y)|_{y=\alpha x} = \alpha E(\alpha x).
\]

That is, using now the conventional notation \( e^{\alpha x} \) for \( E(\alpha x) \), we find that \( f(x) = e^{\alpha x} \) is a solution of the DE:

\[
f' = \alpha f.
\]

A basic question is: are there any other solutions? The answer is yes, but any other solution differs from this one simply by multiplication by a constant:

3
Proposition. If \( g(x) \) is any differentiable function satisfying \( g' = \alpha g \) (where \( \alpha \in \mathbb{R} \) is a constant) and \( f(x) = e^{\alpha x} \), then \( g(x) = Cf(x) \) for some constant \( C \in \mathbb{R} \).

Proof. Since \( f(x) > 0 \), we apply the ‘quotient rule’ to the function \( g(x)/f(x) \):

\[
\left( \frac{g}{f} \right)' = \frac{gf' - g f'}{g^2} = \frac{\alpha gf - g\alpha f}{g^2} = 0,
\]

so \( g/f \) is a constant \( C \).

This proposition says that the ‘general solution’ to the DE \( f' = \alpha f \) is given by:

\[
f(x) = Ce^{\alpha x},
\]

where \( C \) is an arbitrary constant. Substituting \( x = 0 \) in this relation, we find \( C = f(0) \), or:

\[
f(x) = f(0)e^{\alpha x}.
\]

The ‘physical interpretation’ of this conclusion is the following: if \( f(t) \) denotes a positive quantity (mass of a sample, population, etc.) which changes in time, \( f'(t)/f(t) \) is the ‘instantaneous relative growth rate’: the rate of change at time \( t \) divided by the value of \( f \) at time \( t \). Saying \( f' = \alpha f \) means the ‘relative growth rate’ is constant in time, equal to \( \alpha \). The conclusion is that, if \( \alpha > 0 \), the value of \( f \) increases exponentially fast while, if \( \alpha < 0 \), \( f \) decreases to zero exponentially fast. These are the only possible ‘long term behaviors’ under a ‘constant relative growth rate’.

EXERCISES

1. Let \( f(x) = e^{\alpha x} + L \) (for constants \( \alpha, L \)). What is the ‘first-order differential equation’ solved by \( f' \)? (The variable \( x \) should not appear in the equation). What is the ‘general solution’ of this DE? (Hint: consider the equation solved by \( g(x) = f(x) - L \).

(Interpretation: The DE has a constant ‘source term’ (a contribution to the rate of change independent of the value of \( f \)). If \( \alpha < 0 \), the solution ‘stabilizes’ to \( L \) for large \( t \) (decreases to \( L \) if \( f(0) > L \), increases to \( L \) if \( f(0) < L \), where \( L > 0 \) if the source term is positive, \( L < 0 \) if the source term is negative. Draw some graphs!)
2. The ‘hyperbolic sine’ and ‘hyperbolic cosine’ functions are defined (respectively) by:

\[ \sinh(x) = \frac{1}{2}(e^x - e^{-x}), \quad \cosh(x) = \frac{1}{2}(e^x + e^{-x}). \]

(i) Show that both functions are solutions of the ‘second-order linear DE’:

\[ f'' = f. \]

(ii) Find two solutions of the DE \( f'' = \alpha^2 f \), where \( \alpha \in \mathbb{R} \). *(Hint: replace \( x \) by \( \alpha x \) in (i)).*

3. The ‘hyperbolic tangent’ and ‘hyperbolic cotangent’ are defined by:

\[ \tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \quad \coth(x) = \frac{\cosh(x)}{\sinh(x)}. \]

Draw their graphs. Show that both are solutions of the ‘non-linear first-order DE’:

\[ f' + f^2 - 1 = 0. \]

What is the non-linear first-order DE solved by \( \tanh(\alpha x), \coth(\alpha x) \)?

4. The functions \( \tan(\alpha x) \) and \( \cot(\alpha x) \) (the usual tangent and cotangent, evaluated at \( \alpha x \)) solve non-linear first-order DEs similar to that stated in problem 3. Find them.

Remark: The family of DE: \( f' \pm f^2 + k = 0 \), where \( k \in \mathbb{R} \), is known as ‘Ricatti equations’. It is one of the very few non-linear DE admitting ‘explicit closed-form solutions’.