SUMMARY AND CONTEXT OF RESEARCH PUBLICATIONS- Alex Freire (2012)

My research area is Geometric Analysis, with current emphasis on geometric evolution equations and differential-geometric problems motivated by General Relativity. My earliest contributions (1982-1992) were in smooth ergodic theory and analysis on non-compact Riemannian manifolds.

In what follows I introduce the mathematical context of each of my research publications and describe the contributions therein. They are sorted into five groups, in reverse chronological order. (The numbers refer to my publication list.)

1.Geometric problems motivated by General Relativity (2010-present)

In his proof of the three-dimensional Riemannian Penrose inequality with multiple horizons, H. Bray used a mass-capacity inequality in order to prove the monotonicity of the ADM mass along a conformal flow of metrics. His proof used a reflection argument and the Positive Mass Theorem. For connected boundaries, but still in dimension three, Bray and Miao gave a proof using the monotonicity of the Hawking mass along inverse mean curvature flow. For conformally flat manifolds in all dimensions, a mass-capacity inequality was proved by F. Schwartz (2010), with a non-sharp constant.

In [19] we obtain, for conformally flat, asymptotically flat manifolds with minimal boundary, the sharp result that the mass is bounded from below by the L2 capacity of the boundary. As a by-product of our proof we obtained (using inverse mean-curvature flow in Euclidean space) generalizations of classical geometric inequalities for Euclidean domains. First, generalizing an inequality of Polya-Szego for convex domains, we show that for mean-convex hypersurfaces the capacity is bounded above by total mean curvature (normalized to unity on the sphere). Secondly, for mean-convex outer-minimizing hypersurfaces we show that this normalized total mean curvature is bounded below by the normalized area (raised to a suitable power), generalizing a classical result of Aleksandrov-Fenchel.

We continue to work on questions motivated by this work. Extending the geometric inequalities to submanifolds of the sphere is the subject of current work. More generally, our work sheds light on the interaction between geometric flows and mass-type invariants in all dimensions. This should have implications to understanding Penrose-type inequalities for Riemannian manifolds with boundary in higher dimensions.

2. Mean Curvature Flow and Networks (c. 1999- present.)

Paper [15] was motivated, on the one hand, by work of R. Ye and of G. Huisken-S.T.Yau on foliations of asymptotically flat ends by constant mean curvature spheres; and, on the other, by work of Alikakos and collaborators on the mean-curvature motion of small spheres on the boundary of a domain arising as sharp-interface limits of (mass-preserving) Cahn-Hilliard models. We consider the motion of hypersurfaces in Riemannian manifolds with normal velocity given by the mean curvature, minus its average over the hypersurface (corresponding to preservation of the enclosed volume). Earlier work of G. Huisken (late 1980s) established global existence under strong pinching conditions on the curvature tensor of the hypersurface, and observed that singularities develop in general, even for convex initial hypersurfaces on standard spheres. We establish global existence on general manifolds, replacing Huisken's pinching conditions by the assumption that the initial hypersurface is a `small' geodesic sphere. In addition, we prove convergence at infinite time to a constant mean curvature sphere (under a generic condition on the ambient metric). The techniques involve a combination of the maximal

regularity theory approach to abstract evolution equations in Banach spaces (Da Prato-Grisvard-Angenent), a barycentric decomposition of the motion to cancel the bad part of the spectrum of the linearization, and detailed expansions of the curvature tensor in geodesic normal coordinates.

Paper [16] was motivated by work of Mantegazza-Novaga-Tortorelli (2003) dealing with curvature motion of networks in bounded planar domains, with Dirichlet or Neumann boundary conditions. The consideration of stationary solutions (`Steiner networks' with 120 degree triple junctions and orthogonal boundary intersection) is natural in this context, and one might expect that such networks always exist in abundance, at least for smooth, strictly convex bounded planar domains. Surprisingly this is far from being the case. In [16] I show, first, that on such domains only three types of Steiner-Neumann networks may exist. In addition, I show that there exist real-analytic strictly convex domains which do not support Neumann triodes, and such that any neighborhood includes both domains which (stably) support triodes and domains which (stably) do not. Similar statements hold for the other possible networks. Sufficient geometric conditions for existence are also given in each case.

Papers [17] and [18] both address the question of local (or global) existence of mean curvature motion for triple junctions of hypersurfaces in higher dimensions, meeting with constant angles along the manifold of intersection. The emphasis is on classical (Hölder space) solutions. [17] treats the simpler case of a symmetric junction (in all dimensions), which amounts (in the case of graphs) to a free-boundary problem with fixed angle condition for graph mean curvature flow. In [17] I prove local existence by regarding the diffeomorphism pulling back the problem to a fixed domain as governed by its own graph mean curvature-type equation, introducing additional boundary conditions to obtain a well-posed problem. I also include results in the direction of global existence (contraction to a point in finite time) for concave graphs: continuation criterion (so far involving also the covariant derivative of mean curvature along the junction), preservation of concavity, finite existence time. In [18] I prove local existence-uniqueness of classical evolution under graph mean curvature motion of general (non-symmetric) triple junctions of two-dimensional graphs, meeting along the curve of intersection with 120-degree angles. The initial configuration is assumed to satisfy a natural compatibility condition (the mean curvatures along the junction add to zero.) This result answered a 16-year question and opened the way to considering global existence and blowup problems for this system.

Extending these results is the subject of ongoing efforts. In particular, my goal is to obtain global existence under geometrically natural conditions, initially for a single graph evolving with contact-angle condition, as in [17]. In addition, in [17] a maximum principle for systems on manifolds with boundary was obtained, and it is natural both to generalize it and to look for applications to other geometric flows.

3. Harmonic Map Flow and Wave Maps (1993-1998 and beyond.)

In the late 1980's, work of Y. Chen-M. Struwe established the existence of global weak solutions of the harmonic map heat flow to general target manifolds, by a penalty approximation method; there was no uniqueness statement in their paper. Work by N. Rivière (1993) established uniqueness for maps from surfaces to spheres, assuming small total energy for the initial data. In [9] I proved uniqueness of weak harmonic map flows to the sphere under the hypothesis `energy weakly non-increasing with time' (still for two-dimensional domains). The work in [9] combined Struwe's `almost regular' global solutions with the extension to the parabolic setting of a method (due to F. Hélein, late 1980s) using Hardy space-BMO duality (and work of S. Müller on regularity for jacobians) to re-write the harmonic map system in `div-curl' form, leading to strong regularity results. Since then several papers have appeared showing that the hypothesis on the

energy in [9] is sharp. In [10] I extended the uniqueness result to maps to general target manifolds. This extension was far from routine, since it involved constructing `global optimal frames' on the target (in the style of Hélein's work), but now adapted to a parabolic map with weak regularity properties. For higher-dimensional domain manifolds the corresponding natural question is bounding the spacetime Hausdorff dimension of the singular set of a weak flow. When the target is a sphere (under a `stationarity' condition on the flow) this was achieved in the 1994 thesis of M. Feldman. An extension of Feldman's result to general target spaces was obtained very recently by R. Moser.

In the mid-1990s, activity on `wave maps' (the hyperbolic analogue of the harmonic map heat flow) picked up, following early work of Christodoulou, Shatah, Struwe and (along different lines) of Klainerman-Machedon. It was natural to wonder what the `div-curl techniques' would lead to in this case, and this led to [12], [13] and [14]. In [12] I obtained global existence of weak solutions of the wave map system to target spaces with locally homogeneous metric (for any domain dimension). The idea was to use a global frame consisting of Killing vector fields, appealing to an equivariant version of the Nash isometric embedding theorem. In [13] and [14] the div-curl structure (combined with `compensated compactness' techniques) was used to obtain bounds on the Hausdorff dimension of the singular set of a weak solution, as well as a convergence result for wave maps to arbitrary targets (under bounded total energy conditions).

4. Harmonic Functions and L2 spectrum of complete, non-compact manifolds (1988-1994).

The line of research into which my Ph.D. thesis [4] fits began with work of M.Anderson and R.Schoen (1983) on positive (or bounded) harmonic functions in a complete, simply connected manifold with sectional curvature bounded between two negative constants. They were able to solve the Dirichlet problem at infinity (for continuous data), and to identify the Martin boundary (minimal positive harmonic functions) with the sphere at infinity. At around the same time, Ballmann, Brin, Eberlein and Spatzier used the dynamics of geodesic flows to classify the universal covers of compact manifolds of non-positive curvature (as products of symmetric spaces and `rank one' manifolds.) It then became a natural project to try to extend the Anderson-Schoen results to non-positive curvature (perhaps using Ljapunov exponents.) In my thesis, I was able to describe the minimal Martin boundary of a `Hadamard manifold' (complete, simply-connected, non-positive sectional curvature) in terms of those of its irreducible factors. In the process of writing it up for publication, I discovered a much more general result ([5]), applying to general Riemannian products of complete, non-compact manifolds with Ricci curvature bounded below, in particular the following surprising splitting theorem: a bounded harmonic function on the product of two such manifolds (that is, harmonic for the sum of the Laplace-Beltrami operators) is separately harmonic along each factor. Since then, this line of work has been developed by the probability community; experts on Brownian motion on manifolds and its discretizations have given probabilistic proofs of the result, and examples showing the hypotheses are sharp.

Papers [6], [7] and [8] were developed (jointly with J. Escobar) during 1990-94. The original motivation was a problem on S-T. Yau's 1980 list: a complete, noncompact manifold of non-negative sectional curvature has no L2 eigenfunctions (for the Laplace-Beltrami operator), i.e. the point spectrum is trivial; this remains open. Our joint work [6] verified the conjecture under a quadratic curvature decay condition, both for manifolds and for (properly embedded) hypersurfaces in Euclidean space. In [7] we used similar methods to address the L2 Hodge theory (L2 point spectrum on differential forms and L2 harmonic forms) on the same class of manifolds; under quadratic curvature decay we proved vanishing theorems for the spectrum and the L2

cohomology (in the negative curvature case, related work is due to R. Mazzeo.) [8] extends the results of [7] to L2 harmonic forms with values in vector bundles.

5. Geometric Ergodic Theory (1982-1991)

[1] was motivated by work of A.Manning (Annals 1979) relating the topological entropy of the geodesic flow with the exponential growth rate of the volume of geodesic balls in the universal cover, for compact manifolds of non-positive sectional curvature. In the first part of the paper we extended that result to the (much broader) class of compact manifolds without conjugate points, by a different method. In the second part, we used the theory of Ljapunov exponents to give estimates from above and below for the measure-theoretic entropy of the geodesic flow (with Liouville measure) in terms of integrals of Ricci curvature (over the unit tangent bundle) or scalar curvature (over the manifold). The main difficulty was that, without curvature assumptions, the Ljapunov exponents cannot be estimated pointwise using comparison for the matrix Riccatti equation, only in an average sense.

[2] introduces a very simple geometric construction of an invariant probability for a general rational map of the Riemann sphere; additionally, we gave conditions under which this measure is ergodic and Bernouilli. Similar results were obtained independently by M.Ljubich.

[3] dates from my post-doctoral years at Stanford. The motivation for the work was extending to foliations with transverse invariant measure classical results of J. Cheeger and S.-T. Yau for complete manifolds of non-negative Ricci curvature (finite volume implies compact, Liouville theorems, the splitting theorem.) The main results are that if such a foliation has complete leaves of non-negative Ricci curvature, almost every leaf must be flat; and a non-negative, leafwise subharmonic function must be constant. As applications we gave Liouville and rigidity theorems for complete, quasi-periodic Riemannian metrics.