Exam 4

You must upload the solutions to this exam by 11:59pm on *Friday* 08/09. Since this is a take home, I want all your solutions to be neat and well written.

You can look at class discussions on Cocalc and *our* book only (*except* for the hints to exercises in the back of the book)! You *cannot* look at our videos, solutions posted by me or *any* other references (including the Internet) without my previous approval. Also, of course, you cannot discuss this with *anyone*!

You can use a computer only to check your answers, as **you need to show work in all questions**.

1) [50 points] Let $\sigma, \tau \in S_{11}$ be given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 5 & 8 & 10 & 4 & 11 & 6 & 2 & 7 & 1 & 3 & 9 \end{pmatrix} \quad \text{and} \quad \tau = (1\ 5\ 10)(3\ 11\ 2\ 4\ 7)(6\ 8\ 9).$$

(a) Write the complete factorization of σ into disjoint cycles.

(b) Write τ is matrix form.

(c) Compute τ^{-1}. [Your answer *must* be in *disjoint cycles form*!]

(d) Compute $\sigma\tau$. [Your answer *must* be in *disjoint cycles form*!]

(e) Compute $\sigma\tau\sigma^{-1}$. [Your answer *must* be in *disjoint cycles form*!]

(f) Write τ as a product of transpositions.

(g) Compute $\text{sign}(\tau)$.

(h) Compute $|\tau|$.

(i) Give an element $\alpha \in S_{11}$, with $\alpha \neq 1$, such that $\alpha \cdot \tau = \tau \cdot \alpha$ [i.e., α must commute with τ].

(j) Give an element $\beta \in S_{11}$ such that $\beta \cdot \tau \neq \tau \cdot \beta$ [i.e., β does *not* commute with τ]. [**Hint:** You need $\beta \cdot \tau \cdot \beta^{-1} \neq \tau$.]

2) [15 points] Let G be an *Abelian* group. Define then:

$$\text{Tor}(G) = \{x \in G \ : \ x^n = e \text{ for } \textit{some } n \in \mathbb{Z}_{>0}\}.$$

[Here we are using the multiplicative notation and e is the identity element, which we could also denote simply by "1".] Prove that $\text{Tor}(G)$ is a subgroup of G. *Make clear where you use the fact the G is Abelian!* [If you never do, say so.]

[**Careful:** Different x's in $\text{Tor}(G)$ might have different powers that give e, like maybe $x_1^5 = e$, while $x_2^{71} = e$. So, there might not be a common power n that works for every $x \in \text{Tor}(G)$.]

3) The sets below are *not* groups. Justify why not.

(a) [10 points] The set O of all odd integers and 0 with addition. [So, $O = \{0, 1, -1, 3, -3, 5, -5, \ldots\}$.]

(b) [10 points] $S = \left\{ \begin{pmatrix} 2 & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \ : \ 2d - bc \neq 0 \right\}$ with the multiplication of matrices.

4) [15 points] Show that

$$S_3 = \{1, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$$

is not cyclic, but every proper subgroup [i.e., subgroup different of S_3 itself] is cyclic.