Exam 3

1) [10 points] Below is the *Euclidean Algorithm* performed for $f = 2x^8 + 2x^6 + x^5 + 2x^4 + x^3 + 1$ and $g = 2x^5 + 2x^4 + x^3 + 1$ in $\mathbb{F} - 3[x]$. Find the missing polynomials $h_1(x)$, $h_2(x)$, and $h_3(x)$ and the GCD of f and g.

[**Hint:** Be careful with the GCD!]

$$
\begin{aligned}
2x^8 + 2x^6 + x^5 + 2x^4 + x^3 + 1 &= (2x^5 + 2x^4 + x^3 + 1) \cdot (x^3 + 2x^2 + 1) + (2x^3 + x^2) \\
2x^5 + 2x^4 + x^3 + 1 &= (2x^3 + x^2) \cdot h_1(x) + h_2(x) \\
2x^3 + x^2 &= h_3(x) \cdot (x + 2) + (2x + 1) \\
2x^2 + 1 &= (2x + 1) \cdot (x + 1) + 0
\end{aligned}
$$

Solution. Performing the long division [from the second line] of $2x^5 + 2x^4 + x^3 + 1$ by $2x^3 + x^2$, we get quotient $h_1 = x^2 + 2x + 1$ and remainder $h_2 = 2x^2 + 1$.

Then, on the next line we need to divide the previous divisor by the previous remainder, and so $h_2 = h_2 = 2x^2 + 1$.

Now the GCD is the *monic version* of the last non-zero remainder, i.e., the monic version of $2x + 1$, which is $2 \cdot (2x + 1) = x + 2$. □

2) [20 points] Consider the factorization into primes/irreducibles of the following polynomials in $\mathbb{Q}[x]$:

$$
\begin{aligned}
f &= x^2 \cdot (x - 2) \cdot (x^2 - x + 2)^3, \\
g &= x^3 \cdot (x - 1)^5 \cdot (x^2 - 3x + 1)^2 \cdot (x^2 - x + 2).
\end{aligned}
$$

Give the factorization into primes/irreducibles of *all* monic common divisors of f and g.

[**Hint:** There are 6 monic common divisors, including 1 and the GCD. Also, what is the relation between a common divisor and the GCD?]

Solution. First observe that the factors in the given factorizations are irreducible.

A common divisor is a divisor of the GCD, which is $x^2 \cdot (x^2 - x - 2)$ by Proposition 3.86. And, a common divisor So, they are:

$$1, x, x^2, (x^2 - x + 2), x \cdot (x^2 - x + 2), x^2 \cdot (x^2 - x + 2).$$

□

3) [20 points] Give an example of a *non-zero* polynomial f in $\mathbb{F}_5[x]$, such that $f([0]) = f([1]) = f([2]) = f([3]) = f([4]) = [0]$.

Solution. Take $f = x \cdot (x-1) \cdot (x-2) \cdot (x-3) \cdot (x-4) = x^5 - x$. $\square$

4) Examples:

(a) [10 points] Give an example of an domain R that is a subring of $\mathbb{F}_3(x, y)$ such that R is not a field.

Solution. $\mathbb{F}_3[x]$, $\mathbb{F}_3[y]$, and $\mathbb{F}_3[x, y]$, for instance, all work. $\square$

(b) [10 points] Give an example a ring R that is infinite, non-commutative, and for all $a \in R$, we have that $2a = 0$.

Solution. $M_2(\mathbb{F}_2[x])$ works. $\square$

5) Determine if the polynomials below are irreducible or not in the corresponding polynomial ring. *Justify each answer!*

(a) [5 points] $f = x^2 + x + 1$ in $\mathbb{R}[x]$ (*not* in $\mathbb{Q}[x]$).

Solution. The roots of f [using quadratic formula] are $(-1 \pm \sqrt{2}i)/2 \notin \mathbb{R}$. Since the degree is 2 and there are no roots in $\mathbb{R}$, f is irreducible. $\square$

(b) [5 points] $f = 6x^3 - x^2 + x + 7645274672646$ in $\mathbb{Q}[x]$.

Solution. Reduce modulo 5: $\bar{f} = x^3 - x^2 + x + 1 \in \mathbb{F}_5[x]$. Now $\bar{f}([0]) = [1] \neq 0$, $\bar{f}([1]) = [2] \neq 0$, $\bar{f}([2]) = [7] = [2] \neq 0$, $\bar{f}([3]) = [2] \neq 0$, $\bar{f}([4]) = [3] \neq 0$. Since $\deg(\bar{f}) = 3$ and it has no roots, it is irreducible in $\mathbb{F}_5[x]$, and thus f is irreducible in $\mathbb{Q}[x]$. $\square$

(c) [5 points] $f = 7082x^5 - 10000x^4 + 32005x^3 - 37695x + 6000010$ in $\mathbb{Q}[x]$.

Solution. Since $6000010 = 1200002 \cdot 5$, we have that $25 \nmid 6000010$. Then, Eisenstein's Criterion for $p = 5$ tells us that it is irreducible. $\square$

(d) [5 points] $f = x^3 - 2019x^2 + 2019^2 x - 1$ in $\mathbb{Q}[x]$.

Solution. The only possible roots are ± 1 by the rational root test. But $f(1) = 2019^2 - 2019 > 0$ and $f(-1) = -2019 - 2019^2 - 2 < 0$. So there is no root in $\mathbb{Q}$. Since the degree is 3, we have that f is irreducible. $\square$

(e) [5 points] $f = x^7 - \sqrt{2}\, x^3 - \pi\, x^2 + \pi^{\sqrt{2}}$ in $\mathbb{C}[x]$ (*not* in $\mathbb{Q}[x]$).

Solution. Since the degree is not 1, f is irreducible [as it has a root, since $\mathbb{C}$ is *algebraically closed*]. $\square$

(f) [5 points] $f = x^{2020} - 4$ in $\mathbb{Q}[x]$.

Solution. $f = (x^{1010} - 2)(x^{1010} + 2)$, so reducible. $\square$