Exam 3

You must upload the solutions to this exam by 11:59pm on Sunday 08/04. Since this is a take home, I want all your solutions to be neat and well written.

You can look at class discussions on Cocalc and *our* book only (*except* for the hints to exercises in the back of the book)! You *cannot* look at solutions posted by me or *any* other references (including the Internet) without my previous approval. Also, of course, you cannot discuss this with *anyone*!

You can use a computer only to check your answers, as **you need to show work in all questions**.

1) [10 points] Below is the *Euclidean Algorithm* performed for $f = 2x^8 + 2x^6 + x^5 + 2x^4 + x^3 + 1$ and $g = 2x^5 + 2x^4 + x^3 + 1$ in $\mathbb{F}_3[x]$. Find the missing polynomials $h_1(x)$, $h_2(x)$, and $h_3(x)$ and the GCD of f and g.

[**Hint:** Be careful with the GCD!]

$$\begin{aligned}
2x^8 + 2x^6 + x^5 + 2x^4 + x^3 + 1 &= (2x^5 + 2x^4 + x^3 + 1) \cdot (x^3 + 2x^2 + 1) + (2x^3 + x^2) \\
2x^5 + 2x^4 + x^3 + 1 &= (2x^3 + x^2) \cdot h_1(x) + h_2(x) \\
2x^3 + x^2 &= h_3(x) \cdot (x + 2) + (2x + 1) \\
2x^2 + 1 &= (2x + 1) \cdot (x + 1) + 0
\end{aligned}$$

2) [20 points] Consider the factorization into primes/irreducibles of the following polynomials in $\mathbb{Q}[x]$:

$$\begin{aligned}
f &= x^2 \cdot (x - 2) \cdot (x^2 - x + 2)^3, \\
g &= x^3 \cdot (x - 1)^5 \cdot (x^2 - 3x + 1)^2 \cdot (x^2 - x + 2).
\end{aligned}$$

Give the factorization into primes/irreducibles of *all* monic common divisors of f and g.

[**Hint:** There are 6 monic common divisors, including 1. Also, what is the relation between a common divisor and the GCD?]

3) [20 points] Give an example of a *non-zero* polynomial f in $\mathbb{F}_5[x]$, such that $f([0]) = f([1]) = f([2]) = f([3]) = f([4]) = [0]$.

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4) Examples:

(a) [10 points] Give an example of a domain R that is a subring of $\mathbb{F}_3(x, y)$ but such that R is not a field.

(b) [10 points] Give an example a ring R that is infinite, non-commutative, and for all $a \in R$, we have that $2a = 0$ [which is the same as to say that $a + a = 0$].

5) Determine if the polynomials below are irreducible or not in the corresponding polynomial ring. *Justify each answer!*

(a) [5 points] $f = x^2 + x + 1$ in $\mathbb{R}[x]$ (*not* in $\mathbb{Q}[x]$).

(b) [5 points] $f = 6x^3 - x^2 + x + 7645274672646$ in $\mathbb{Q}[x]$.

(c) [5 points] $f = 7082x^5 - 10000x^4 + 32005x^3 - 37695x + 6000010$ in $\mathbb{Q}[x]$.

(d) [5 points] $f = x^3 - 2019x^2 + 2019^2x - 1$ in $\mathbb{Q}[x]$.

(e) [5 points] $f = x^7 - \sqrt{2}\,x^3 - \pi\,x^2 + \pi^{\sqrt{2}}$ in $\mathbb{C}[x]$ (*not* in $\mathbb{Q}[x]$).

(f) [5 points] $f = x^{2020} - 4$ in $\mathbb{Q}[x]$. [**Hint:** $4 = 2^2$.]