Exam 2

You must upload the solutions to this exam by 11:59pm on *Sunday* 07/28. Since this is a take home, I want all your solutions to be neat and well written.

You can look at class discussions on Cocalc and *our* book only (*except* for the hints to exercises in the back of the book)! You *cannot* look at our videos, solutions posted by me or *any* other references (including the Internet) without my previous approval. Also, of course, you cannot discuss this with *anyone*!

You can use a computer only to check your answers, as **you need to show work in all questions**.

1) [15 points] Find all the units of $\mathbb{I}_{14}$ and for each unit, find its inverse.

2) [20 points] For all examples below, check *all* the boxes that apply [no need to justify]:

(a) $\mathbb{N} = \{0, 1, 2, \ldots\}$: ☐ non-commutative ring, ☐ commutative ring, ☐ domain, ☐ field.

(b) $\mathbb{R}$: ☐ non-commutative ring, ☐ commutative ring, ☐ domain, ☐ field.

(c) $\mathbb{I}_5[x]$: ☐ non-commutative ring, ☐ commutative ring, ☐ domain, ☐ field.

(d) $\mathbb{I}_6[x]$: ☐ non-commutative ring, ☐ commutative ring, ☐ domain, ☐ field.

(e) $M_2(\mathbb{Q})$ [i.e., 2×2 matrices with entries in $\mathbb{Q}$]: ☐ non-commutative ring, ☐ commutative ring, ☐ domain, ☐ field.

3) [15 points] Give the prime field of the following fields [no need to justify]:

(a) $\mathbb{Q}$

(b) $\mathbb{R}(x)$

(c) $\mathbb{F}_p(x, y)$ [Note that $\mathbb{F}_p(x, y)$ is the field of rational functions in two variables. You can see if as the field of fractions of $\mathbb{F}_p(x)[y]$, i.e., $\mathbb{F}_p(x)(y)$.]

4) Let $R = \mathbb{Z}[\sqrt{3}] = \{a + b\sqrt{3} \ : \ a, b \in \mathbb{Z}\}$.

(a) [10 points] Is R a commutative ring? [Justify!]

(b) [5 points] Is R an integral domain? [Justify!]

(c) [5 points] Is R a field? [Justify!]

5) Let F be the field of fractions of the Gaussian integers $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$. [Remember that i is the complex number with $i^2 = -1$ and that $\mathbb{Z}[i]$ is a domain.]

(a) [5 points] Is $\dfrac{2-3i}{3+2i} = \dfrac{-1}{i}$ in F? [Show your computations.]

(b) [10 points] Let $\alpha = \dfrac{1}{2+i}$ and $\beta = \dfrac{1+i}{2-i}$. Compute $\alpha + \beta$ and $\alpha \cdot \beta$ [in F]. [Show work!]