# Midterm Solutions

M460 – Geometry

**1. Problem 2.12(c):** Give a model of incidence geometry in which we have three [distinct] lines $l$, $m$ and $n$ such that $l \parallel m$ and $m \parallel n$, but $l \nparallel n$. [Give a concrete example of such lines in the given model!]

*Solution.* One can use the upper half plane model. Consider the [upper half of] $x = 0$, $(x-3)^2 + y^2 = 1$ and $(x-0)^2 + y^2 = 1$. The first does not intersect the second, the second does not intersect the third, but the first does intersect the third. $\square$

**2.** We will deal with the model of incidence geometry given by $\mathbb{Q}^2 = \{(x,y) \in \mathbb{R}^2 : x, y \in \mathbb{Q}\}$. Points are the elements of $\mathbb{Q}^2$ [i.e., points of $\mathbb{R}^2$ with both coordinates in $\mathbb{Q}$] and lines are given by the solutions [in $\mathbb{Q}^2$] of equations of the form $ax + by + c = 0$, where $a, b, c \in \mathbb{Q}$. [You can assume that this is a model of incidence geometry without proving it.] Prove that $\mathbb{Q}^2$ does not satisfy the Congruence Axiom 1.

*Proof.* Consider $A = (0,0)$, $B = (1,1)$ and $C = (1,0)$. Then $\bar{AB} = \sqrt{2}$. If we have $P \in \overrightarrow{AC}$ such that $AP \cong AB$, then $P$ must be $(\sqrt{2}, 0)$, as this is the only point in $\mathbb{R}^2$ with this property, but $(\sqrt{2}, 0) \notin \mathbb{Q}^2$. $\square$

**3. Problem 2.15(a):** Let $M$ be a model of incidence geometry in which every line has at least three distinct points in it. We will show that there are four point in this model with no three of them on the same line.

(i) Start with three points, say $A$, $B$, $C$, not on the same line [by I-3]. Thus, one can easily prove that lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$ and $\overleftrightarrow{AC}$ are distinct. [You can use it without proving it!]

(ii) By hypothesis, line $\overleftrightarrow{AB}$ has a third point, say $B'$, and line $\overleftrightarrow{AC}$ has a third point, say $C'$.

(iii) Consider the line $\overleftrightarrow{B'C'}$. By hypothesis there is a third point on it, say $D$.

[Draw a picture!]

(a) Prove that $\overleftrightarrow{B'C'} \neq \overleftrightarrow{AB}$. [In a similar way, you could prove that $\overleftrightarrow{B'C'} \neq \overleftrightarrow{BC}, \overleftrightarrow{AC}$. **You may use this fact in the next parts!**]

*Proof.* Suppose they are equal. Then, $A, C' \in \overleftrightarrow{AB}$. Also, $A, C' \in \overleftrightarrow{AC}$, by (ii). Since $C \neq C'$, by (ii) again, we have, by I-1, that $\overleftrightarrow{AB} = \overleftrightarrow{AC}$, which contradicts (i). Therefore, they must be different. □

(b) Prove that $A$, $B$ and $D$ are not on the same line. [In a similar manner you could prove that not three among $A$, $B$, $C$ and $D$ are in the same line, finishing the proof of 2.15(a).]

*Proof.* Suppose they are on the same line, namely $\overleftrightarrow{AB}$. Then, $D \in \overleftrightarrow{AB}$. Also, by (ii), we have that $B' \in \overleftrightarrow{AB}$. But, moreover, we have that $D, B' \in \overleftrightarrow{B'C'}$, by (iii), which also gives us that $D \neq B'$. Therefore, by I-1, we must have that $\overleftrightarrow{B'C'} = \overleftrightarrow{AB}$, which is a contradiction by part (a). □

**4. Problem 3.9:** Given a line $l$, a point $A$ on $l$ and a point $B$ not on $l$, prove that every point of $\overrightarrow{AB}$ except $A$ is on the same side of $l$ as $B$. [**Hint:** Use an RAA argument. You may also use the fact that if $X$, $Y$ and $P$ are colinear, then $P \notin \overrightarrow{XY}$ is equivalent to $P * X * Y$.]

*Proof.* Suppose that $P \neq A$ and $P \in \overrightarrow{AB}$, and $P$ and $B$ on opposite sides of $l$. By definition, $P \neq B$ and there is $Q \in l$ such that $PB \cap l = \{Q\}$. Since $P, B \in \overrightarrow{AB}$ and $P \neq B$, by I-1, we have that $\overleftrightarrow{PB} = \overleftrightarrow{AB}$.

Note that since $B \in \overleftrightarrow{AB}$, but $B \notin l$ [by hypothesis], we must have that $l \neq \overleftrightarrow{AB}$. Moreover, since $A \in l$ and $A \in \overleftrightarrow{AB}$, by I-1 we must have that $\overleftrightarrow{AB} \cap l = \{A\}$, or else we would have another point and the lines would be equal, which is a contradiction.

Now, $\{Q\} = PB \cap l \subseteq \overleftrightarrow{PB} \cap l = \overleftrightarrow{AB} \cap l = \{A\}$. Therefore, $Q = A$ and so we have $P * A * B$. But, mentioned in the hint, since $P * A * B$, we have that $P \notin \overrightarrow{AB}$, which is a contradiction. □