1) Let $\mathbf{v} = (1,2,0,3)$ and $\mathbf{w} = (0,1,2,1)$. Find:

(a) [5 points] the cosine of the angle θ between $\mathbf{v}$ and $\mathbf{w}$ [no need to simplify the number]:

Solution.

$$\cos(\theta) = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|} = \frac{5}{\sqrt{14}\sqrt{6}}.$$

□

(b) [5 points] the projection of $\mathbf{v}$ on the direction of $\mathbf{w}$:

Solution.

$$\mathrm{proj}_{\mathbf{w}}(\mathbf{v}) = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \frac{5}{6}(0,1,2,1) = \left(0, \frac{5}{6}, \frac{5}{3}, \frac{5}{6}\right).$$

□

(c) [5 points] the component of $\mathbf{v}$ orthogonal to $\mathbf{w}$:

$$\mathbf{v} - \mathrm{proj}_{\mathbf{w}}(\mathbf{v}) = (1,2,0,3) - \left(0, \frac{5}{6}, \frac{5}{3}, \frac{5}{6}\right) = \left(1, \frac{7}{6}, -\frac{5}{3}, \frac{13}{6}\right).$$

2) [10 points] Let $\mathbf{v}_1 = (1,2,1)$, $\mathbf{v}_2 = (0,1,1)$, and $S = \{\mathbf{v}_1, \mathbf{v}_2\}$. Is $\mathbf{v} = (2,3,1)$ in $\mathrm{span}(S)$? How about $\mathbf{w} = (1,0,0)$? For each affirmative answer, write the corresponding vector a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Solution. We have

$$\left[\begin{array}{cc|c|c} 1 & 0 & 2 & 1 \\ 2 & 1 & 3 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

□

So, $\mathbf{v} \in \mathrm{span}(S)$ and $\mathbf{v} = 2\mathbf{v}_1 - \mathbf{v}_2$, but $\mathbf{w}$ is not.

3) [15 points] Is $S_1 = \{1+x, -1+x\}$ a basis of P_1? [Remember that P_1 is the vector space of polynomials of degree at most 1.] How about $S_2 = \{1, x, 1+x\}$?

Solution. S_2 is not a basis, as $\dim P_1 = 2$ and S_2 has three vectors. Since S_1 has the correct number of vectors [equal to the dimension] it suffices to check if S_1 is linearly independent. But, $k_1(1+x) + k_2(-1+x) = 0$ gives the following linear system:

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since the coefficient matrix has determinant $2 \neq 0$, the system has only the trivial solution. Therefore, S_1 is linearly independent and thus a basis.

□

4) Change of basis:

(a) [10 points] Let $B = \{(2,1), (1,1)\}$ and $B' = \{(1,1), (0,1)\}$. Give the transition matrix $P_{B \to B'}$.

Solution.

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right].$$

So,

$$P_{B \to B'} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}.$$

□

(b) [5 points] Let B and B' be bases of a vector space V, with transition matrix

$$P_{B \to B'} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}.$$

Then, if $[\mathbf{v}]_B = (-1, 2)$, find $[\mathbf{v}]_{B'}$.

Solution.

$$[\mathbf{v}]_{B'} = P_{B \to B'} \cdot [\mathbf{v}]_B = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

□

5) Let $\mathbf{v}_1 = (1, 0, 3, -1, 0)$, $\mathbf{v}_2 = (0, 1, 2, 1, 0)$, and $\mathbf{v}_3 = (2, -1, 4, -3, 1)$. Find a basis for the orthogonal complement of $\text{span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$.

Solution. The orthogonal complement is the nullspace of the matrix with the $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ as rows. Thus, we have:

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 & -1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 2 & -1 & 4 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & -1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & -1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solving we get $(x_1, x_2, x_3, x_4, x_5) = (-3s+t, -2s-t, s, t, 0) = s(-3, -2, 1, 0, 0)+t(1, -1, 0, 1, 0)$. So, a basis is: $\{(-3, -2, 1, 0, 0), (1, -1, 0, 1, 0)\}$. $\square$

6) Let

$$\begin{aligned}
\mathbf{v}_1 &= (-3, -3, -7, -34, -11, 3, -33),\\
\mathbf{v}_2 &= (2, 2, 4, 20, 6, -2, 20),\\
\mathbf{v}_3 &= (1, 1, 2, 10, 3, -1, 10),\\
\mathbf{v}_4 &= (2, 2, 5, 24, 8, -1, 21),\\
\mathbf{v}_5 &= (-1, -1, -4, -18, -7, -1, -12),
\end{aligned}$$

and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, and $V = \operatorname{span}(S)$. Given that

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \\ \mathbf{v}_5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 & -1 & 0 & 2 \\ 0 & 0 & 1 & 4 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

answer the questions below. [You do *not* need to justify any of the items below.]

(a) [4 points] What are the dimension of V and $V^\perp$ [the orthogonal complement of V in $\mathbb{R}^7$]?

Solution. They are 3 [rank of the first matrix] and 4 [nullity of the first matrix] respectively. □

(b) [4 points] Find a basis for V made of elements of S.

Solution. We can take $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$. □

(c) [3 points] Find the coordinates of $\mathbf{v}_2$ and $\mathbf{v}_5$ with respect to the basis you've found in item (b).

Solution. We have $[\mathbf{v}_2]_B = (0, 1, 0)$ and $[\mathbf{v}_5]_B = (0, 3/2, -2)$. □

(d) [3 points] Which vectors from the standard basis of $\mathbb{R}^7$ can you add to the vectors in the basis of V you've in (b) to obtain a basis of all of $\mathbb{R}^7$?

Solution. $\mathbf{e}_2$, $\mathbf{e}_4$, $\mathbf{e}_5$, $\mathbf{e}_7$. □

7) [15 points] Let V be the set of all polynomial of the form $a_0 + a_1 x + (a_0 + a_1)x^2$, where $a_0, a_1 \in \mathbb{R}$. [In other words, polynomials of degree at most two, whose coefficient of x^2 is the sum of the coefficient of x and free coefficient.] Show that V is a vector space.

Solution. Since V is a subset of a vector space [namely, P_2], we only need to check that $0 \in V$ and axiom 0 from the list at the end. Clearly, $0 \in V$, by taking $a_0 = a_1 = 0$.

Also, if $a_0 + a_1 x + (a_0 + a_1)x^2, b_0 + b_1 x + (b_0 + b_1)x^2 \in V$, then:

$$(a_0+a_1x+(a_0+a_1)x^2)+(b_0+b_1x+(b_0+b_1)x^2) = (a_0+b_0)+(a_1+b_1)x+((a_0+b_0)+(a_1+b_1))x^2,$$

and hence the sum is in V.

Finally, if $a_0 + a_1 x + (a_0 + a_1)x^2 \in V$ and $k \in \mathbb{R}$, then:

$$k(a_0 + a_1 x + (a_0 + a_1)x^2) = (ka_0) + (ka_1)x + (ka_0 + ka_1)x^2,$$

and hence the scalar product is in V.

$\square$

Vector Space Axioms

A non-empty set V with a sum and a scalar product is a vector space if it satisfies the following conditions:

0. $\mathbf{u} + \mathbf{v} \in V$ for all $\mathbf{u}, \mathbf{v} \in V$, and $k\mathbf{u} \in V$ for all $\mathbf{u} \in V$ and $k \in \mathbb{R}$;

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$;

2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$;

3. there is $\mathbf{0} \in V$ such that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$;

4. given $\mathbf{u} \in V$, there exists $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$;

5. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in V$ and $k \in \mathbb{R}$;

6. $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$ for all $\mathbf{u} \in V$ and $k, l \in \mathbb{R}$;

7. $k(l\mathbf{u}) = (kl)\mathbf{u}$ for all $\mathbf{u} \in V$ and $k, l \in \mathbb{R}$;

8. $1\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$.