June 28th, 2011

Math 251

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Name: ....

Student ID (last 6 digits): XXX- ....

Midterm 2

You have 60 minutes to complete the exam. Do all work on this exam, i.e., on the page of the respective assignment. Indicate clearly, when you continue your solution on the back of the page or another part of the exam.

Write your name and the last six digits of your student ID number on the top of this page. Check that no pages of your exam are missing. This exam has 7 questions and 11 printed pages (including this one, a page with the vector space axioms, and a page for scratch work in the end).

No books, notes or calculators are allowed on this exam!

Show all work! (Unless I say otherwise.) Correct answers without work will receive **zero**. Also, **points will be taken from messy solutions**.

Good luck!

Question	Max. Points	Score
1	15	
2	10	
3	15	
4	15	
5	15	
6	15	
7	15	
Total	100	

1) Let $\mathbf{v} = (1, 2, 0, 3)$ and $\mathbf{w} = (0, 1, 2, 1)$. Find:

(a) [5 points] the cosine of the angle θ between $\mathbf{v}$ and $\mathbf{w}$ [no need to simplify the number]:

(b) [5 points] the projection of $\mathbf{v}$ on the direction of $\mathbf{w}$:

(c) [5 points] the component of $\mathbf{v}$ orthogonal to $\mathbf{w}$:

2) [10 points] Let $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (0, 1, 1)$, and $S = \{\mathbf{v}_1, \mathbf{v}_2\}$. Is $\mathbf{v} = (2, 3, 1)$ in span(S)? How about $\mathbf{w} = (1, 0, 0)$? For each affirmative answer [if any], write the corresponding vector as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

3) [15 points] Is $S_1 = \{1 + x, -1 + x\}$ a basis of P_1? [Remember that P_1 is the vector space of polynomials of degree at most 1.] How about $S_2 = \{1, x, 1 + x\}$?

4) Change of basis:

(a) [10 points] Let $B = \{(2,1),(1,1)\}$ and $B' = \{(1,1),(0,1)\}$. Give the transition matrix $P_{B \to B'}$.

(b) [5 points] Let B and B' be bases of a vector space V, with transition matrix

$$P_{B \to B'} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}.$$

Then, if $[\mathbf{v}]_B = (-1,2)$, find $[\mathbf{v}]_{B'}$.

5) [15 points] Let $\mathbf{v}_1 = (1, 0, 3, -1, 0)$, $\mathbf{v}_2 = (0, 1, 2, 1, 0)$, and $\mathbf{v}_3 = (2, -1, 4, -3, 1)$. Find a basis for the orthogonal complement of $\text{span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$.

6) Let

$$\begin{aligned}
\mathbf{v}_1 &= (-3,-3,-7,-34,-11,3,-33),\\
\mathbf{v}_2 &= (2,2,4,20,6,-2,20),\\
\mathbf{v}_3 &= (1,1,2,10,3,-1,10),\\
\mathbf{v}_4 &= (2,2,5,24,8,-1,21),\\
\mathbf{v}_5 &= (-1,-1,-4,-18,-7,-1,-12),
\end{aligned}$$

and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, and $V = \operatorname{span}(S)$. Given that

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \\ \mathbf{v}_5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 & -1 & 0 & 2 \\ 0 & 0 & 1 & 4 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

answer the questions below. [You do *not* need to justify any of the items below.]

(a) [4 points] What are the dimension of V and $V^{\perp}$ [the orthogonal complement of V in $\mathbb{R}^7$]?

(b) [4 points] Find a basis for V made of elements of S.

(c) [4 points] Find the coordinates of $\mathbf{v}_2$ and $\mathbf{v}_5$ with respect to the basis you've found in item (b).

(d) [3 points] Which vectors from the standard basis of $\mathbb{R}^7$ can you add to the vectors in the basis of V you've in (b) to obtain a basis of all of $\mathbb{R}^7$?

7) [15 points] Let V be the set of all polynomial of the form $a_0 + a_1x + (a_0 + a_1)x^2$, where $a_0, a_1 \in \mathbb{R}$. [In other words, polynomials of degree at most two, whose coefficient of x^2 is the sum of the coefficient of x and free coefficient.] Show that V is a vector space.

Vector Space Axioms

A non-empty set V with a sum and a scalar product is a vector space if it satisfies the following conditions:

0. $\mathbf{u}+\mathbf{v} \in V$ for all $\mathbf{u}, \mathbf{v} \in V$, and $k\mathbf{u} \in V$ for all $\mathbf{u} \in V$ and $k \in \mathbb{R}$;
1. $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$;
2. $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$;
3. there is $\mathbf{0} \in V$ such that $\mathbf{0}+\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$;
4. given $\mathbf{u} \in V$, there exists $-\mathbf{u} \in V$ such that $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$;
5. $k(\mathbf{u}+\mathbf{v}) = k\mathbf{u}+k\mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in V$ and $k \in \mathbb{R}$;
6. $(k+l)\mathbf{u} = k\mathbf{u}+l\mathbf{u}$ for all $\mathbf{u} \in V$ and $k, l \in \mathbb{R}$;
7. $k(l\mathbf{u}) = (kl)\mathbf{u}$ for all $\mathbf{u} \in V$ and $k, l \in \mathbb{R}$;
8. $1\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$.

Scratch: