1) [10 points] Put the following matrix in *reduced* row echelon form:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 2 \\ 2 & 0 & 4 & 0 & 2 \\ 3 & 1 & 7 & 0 & 5 \\ 0 & 1 & 1 & 1 & 3 \end{bmatrix}.$$

Solution.

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 2 \\ 2 & 0 & 4 & 0 & 2 \\ 3 & 1 & 7 & 0 & 5 \\ 0 & 1 & 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 3 & 1 & 7 & 0 & 5 \\ 0 & 1 & 1 & 1 & 3 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

□

2) [10 points] Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & -2 \\ 1 & -1 & 3 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

Compute $\det(A)$.

Solution.

$$\begin{vmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & -2 \\ 1 & -1 & 3 & 2 \\ 1 & 2 & 1 & 2 \end{vmatrix} = -2 \begin{vmatrix} 1 & 1 & 2 \\ 1 & -1 & 2 \\ 1 & 2 & 2 \end{vmatrix} - 2 \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 3 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= -2 \cdot 0 + -2((-1 + 3 + 0) - (0 + 6 + 1)) = 10.$$

□

3) [40 points] You should be able to answer the following questions *quickly*. Give *short* justifications [or show a little work] unless stated otherwise.

(a) [4 points] What is the dimension of P_5 [where P_5 is the vector space of polynomials of degree at most 5]? [No need to justify.]

Answer. 6 [as $\{1, x, x^2, x^3, x^4.x^5\}$ is a basis]. □

(b) [4 points] Is $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} -2 & -5 \\ 6 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ linearly independent in $M_{2\times 2}$ [i.e., in the vector space of 2×2 matrices]? Justify your answer in one short sentence.

Answer. No, as we have more vectors, namely 5, than the dimension of $M_{2\times 2}$, namely 4. □

(c) [4 points] If $A = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$, then find A^{-1}. [No need to show work.]

Answer. $\frac{1}{3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$. □

(d) [4 points] Give the matrix that represents the rotation by $\pi/2$ about the x-axis, followed by a projection onto the xy-plane in $\mathbb{R}^3$. [No need to justify.]

Answer.

$$\begin{array}{ccccc} \mathbf{e}_1 & \longrightarrow & \mathbf{e}_1 & \longrightarrow & \mathbf{e}_1 \\ \mathbf{e}_2 & \longrightarrow & \mathbf{e}_3 & \longrightarrow & \mathbf{0} \\ \mathbf{e}_3 & \longrightarrow & -\mathbf{e}_2 & \longrightarrow & -\mathbf{e}_2 \end{array}$$

and so,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

□

(e) [4 points] If $A\mathbf{x} = \mathbf{b}$ has no solution, then what can we say about the reduced echelon form of A. [No need to justify.]

Answer. It has a row of zeros. □

(f) [4 points] Let $S, T : \mathbb{R}^4 \to \mathbb{R}^4$ be linear transformations given by

$$T(x_1, x_2, x_3, x_4) = (x_1, x_3, x_2, x_4),$$
$$S(x_1, x_2, x_3, x_4) = (x_1 + x_2, 2x_1 - x_3, 0, x_1 + x_2 + x_4).$$

Give $[T \circ S]$.

Answer. We have

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad [S] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix},$$

and so,

$$[T \circ S] = [T] \cdot [S] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

□

(g) [4 points] Let $\mathbf{v} = (1, 0, 0, 1)$ and $\mathbf{w} = (2, -1, 3, 1)$. Find the component of $\mathbf{w}$ orthogonal to $\mathbf{v}$.

Answer. We have $\operatorname{proj}_{\mathbf{v}} \mathbf{w} = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|^2}\mathbf{v} = \dfrac{3}{2}(1, 0, 0, 1)$, and the orthogonal component is then $(2, -1, 3, 1) - (3/2, 0, 0, 3/2) = (1/2, -1, 3, -1/2)$.

□

(h) [4 points] If A is an invertible matrix, with $A^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$ [note that this is the **inverse** of A, not A!], find the solutions [if any] of $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = (1, 0)$ and $\mathbf{b} = (2, 1)$. [So, there are two systems to solve!]

Answer. Since A is invertible, the solution is $A^{-1}\mathbf{b}$. So, the solutions are $(1, -1)$ and $(4, -3)$. □

(i) [4 points] Let $S = \{(1, 0, 1), (-2, 1, 1), (0, 0, 3)\}$, and $V = \operatorname{span}(S)$. Describe how you would find if $\mathbf{v} = (1, -3, 1)$ is in V. More precisely, set up a system [in matrix form!] and say how solving the system would tell you if $\mathbf{v} \in V$ or not. [The answer is short!]

Answer. We have that $\mathbf{v} \in V$ if, and only if, the system

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

has a solution. □

(j) [4 points] If A is a 5 by 4 matrix of nullity 3, give the rank of A and the nullity of A^{T}. [No need to justify.]

Answer. Since the nullity plus the rank is the number of columns, we have that the rank of A is $4 - 3 = 1$. Since the rank of A is the same as the rank of its transpose, we have that the nullity of A^{T} is $5 - 1 = 4$. □

4) [15 points] Let

$$A = \begin{bmatrix} 3 & 1 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(a) [4 points] Show that the eigenvalues of A are -1 and 3.

Solution. The characteristic equation is given by

$$\det(\lambda I_3 - A) = \begin{vmatrix} (\lambda - 3) & -1 & -3 \\ 0 & (\lambda + 1) & 0 \\ 0 & 0 & (\lambda - 3) \end{vmatrix} = (\lambda - 3)(\lambda + 1)(\lambda - 3) = 0.$$

Thus, the eigenvalues are the solutions $\lambda = 3, -1$. □

(b) [4 points] Find the eigenspace associated to the eigenvalue 3. [Since I gave you the eigenvalue, you can do this part even if you could not do part (a).]

Solution. For the eigenvalue $\lambda = 3$, we need to find the null space of

$$3I_3 - A = \begin{bmatrix} 0 & -1 & -3 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which has reduced echelon form

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, the eigenspace is $\text{span}(\{(1, 0, 0)\})$. □

5) [10 points] Is the set V of all 2 by 2 *diagonal* matrices [with the usual sum and scalar multiplication of matrices] a subspace of $M_{2\times 2}$ [i.e., of the vector space of all 2 by 2 matrices]? [Justify!]

Solution. Yes! Since it is a subspace, we only need to check that it is closed under addition and scalar multiplication.

Let $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \in V$ and $k \in \mathbb{R}$. Then,

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} a+c & 0 \\ 0 & b+d \end{bmatrix}$$

and

$$k \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} ka & 0 \\ 0 & kb \end{bmatrix}.$$

Since both are in V [i.e., both are diagonal], we have that V is a subspace. □

6) [20 points] Let

$$\begin{aligned}\mathbf{v}_1 &= (-3,-3,-7,-34,-11,3,-33),\\ \mathbf{v}_2 &= (2,2,4,20,6,-2,20),\\ \mathbf{v}_3 &= (1,1,2,10,3,-1,10),\\ \mathbf{v}_4 &= (2,2,5,24,8,-1,21),\\ \mathbf{v}_5 &= (-1,-1,-4,-18,-7,-1,-12),\end{aligned}$$

and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, and $V = \operatorname{span}(S)$. Given that

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \\ \mathbf{v}_5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 & -1 & 0 & 2 \\ 0 & 0 & 1 & 4 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

answer the questions below. [You do *not* need to justify any of the items below.]

(a) [4 points] What are the dimension of V and $V^\perp$ [the orthogonal complement of V in $\mathbb{R}^7$]?

Solution. They are 3 and 4 respectively. □

(b) [4 points] Find a basis for V.

Solution. We can take the first three rows of the first matrix in echelon form, say $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ or $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$. □

(c) [4 points] Describe $V^\perp$ as a matrix space [row space, column space, null space] of some matrix. [In other words, fill in the blanks of "$V^\perp$ is the ____ space of the matrix ____."]

Solution. $V^\perp$ is the null space of the first matrix in echelon form [or of the matrix with the $\mathbf{v}_i$'s as rows]. □

(d) [4 points] Find the coordinates of each of the vectors in S with respect to the basis you've found in item (b).

Solution. If we use the basis $B = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, then

$$(\mathbf{v}_1)_B = (-3, -7, 3) \qquad (\mathbf{v}_2)_B = (2, 4, -2)$$
$$(\mathbf{v}_3)_B = (1, 2, -1) \qquad (\mathbf{v}_4)_B = (2, 5, -1)$$
$$(\mathbf{v}_5)_B = (-1, -4, -1).$$

If we use $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$, then

$$(\mathbf{v}_1)_{B'} = (1, 0, 0) \qquad (\mathbf{v}_2)_{B'} = (0, 1, 0)$$
$$(\mathbf{v}_3)_{B'} = (0, 1/2, 0) \qquad (\mathbf{v}_4)_{B'} = (0, 0, 1)$$
$$(\mathbf{v}_5)_{B'} = (0, 3/2, -2).$$

□

(e) [4 points] Which vectors from the standard basis of $\mathbb{R}^7$ you can add to the vectors in the basis of V you've in (b) to obtain a basis of all of $\mathbb{R}^7$?

Solution. $\mathbf{e}_2$, $\mathbf{e}_4$, $\mathbf{e}_5$, $\mathbf{e}_7$. □