July 7th, 2010

Math 251

Luís Finotti
Summer 2010

Name: ....

Student ID (last 6 digits): XXX- ....

Final

You have one hour and a half to complete the exam. Do all work on this exam, i.e., on the page of the respective assignment. Indicate clearly, when you continue your solution on the back of the page or another part of the exam.

Write your name and the last six digits of your student ID number on the top of this page. Check that no pages of your exam are missing. This exam has 6 questions and 12 printed pages (including this one and a page for scratch work in the end).

No books, notes or calculators are allowed on this exam!

Show all work! (Unless I say otherwise.) Even correct answers without work may result in point deductions. Also, **points will be taken from messy solutions**.

Good luck!

Question	Max. Points	Score
1	10	
2	10	
3	40	
4	10	
5	10	
6	20	
Total	100	

1) [10 points] Put the following matrix in *reduced* row echelon form:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 2 \\ 2 & 0 & 4 & 0 & 2 \\ 3 & 1 & 7 & 0 & 5 \\ 0 & 1 & 1 & 1 & 3 \end{bmatrix}.$$

2) [10 points] Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & -2 \\ 1 & -1 & 3 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

Compute $\det(A)$.

3) [40 points] You should be able to answer the following questions *quickly*. Give *short* justifications [or show a little work] unless stated otherwise.

(a) [4 points] What is the dimension of P_5 [where P_5 is the vector space of polynomials of degree at most 5]? [No need to justify.]

(b) [4 points] Is $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} -2 & -5 \\ 6 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ linearly independent in $M_{2\times 2}$ [i.e., in the vector space of 2×2 matrices]? Justify your answer in one short sentence.

(c) [4 points] If $A = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$, then find A^{-1}. If A is not invertible, say so and justify.

(d) [4 points] Give the matrix that represents the rotation by $\pi/2$ about the x-axis, followed by a projection onto the xy-plane. [Both in $\mathbb{R}^3$, of course. No need to justify.]

(e) [4 points] If $A\mathbf{x} = \mathbf{b}$ has no solution, then what can we say about the reduced row echelon form of A. [No need to justify.]

(f) [4 points] Let $S, T : \mathbb{R}^4 \to \mathbb{R}^4$ be linear transformations given by

$$T(x_1, x_2, x_3, x_4) = (x_1, x_3, x_2, x_4),$$
$$S(x_1, x_2, x_3, x_4) = (x_1 + x_2, 2x_1 - x_3, 0, x_1 + x_2 + x_4).$$

Give $[T \circ S]$.

(g) [4 points] Let $\mathbf{v} = (1, 0, 0, 1)$ and $\mathbf{w} = (2, -1, 3, 1)$. Find the component of $\mathbf{w}$ orthogonal to $\mathbf{v}$.

(h) [4 points] If A is an invertible matrix, with $A^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$ [note that this is the **inverse** of A, not A!], find the solutions [if any] of $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = (1, 0)$ and $\mathbf{b} = (2, 1)$. [So, there are two systems to solve!]

(i) [4 points] Let $S = \{(1, 0, 1), (-2, 1, 1), (0, 0, 3)\}$, and $V = \text{span}(S)$. Describe how you would find if $\mathbf{v} = (1, -3, 1)$ is in V. More precisely, set up a system [in matrix form!] and say how solving the system would tell you if $\mathbf{v} \in V$ or not. [The answer is short!]

(j) [4 points] If A is a 5 by 4 matrix of nullity 3, give the rank of A and the nullity of A^{T}. [No need to justify.]

4) [10 points] Let

$$A = \begin{bmatrix} 3 & 1 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(a) [5 points] Show that the eigenvalues of A are -1 and 3.

(b) [5 points] Find the eigenspace associated to the eigenvalue 3. [Since I gave you the eigenvalue, you can do this part even if you could not do part (a).]

5) [10 points] Is the set V of all 2 by 2 *diagonal* matrices $\left[\begin{array}{cc} a & 0 \\ 0 & b \end{array}\right]$ [with the usual sum and scalar multiplication of matrices] a subspace of $M_{2\times 2}$ [i.e., of the vector space of all 2 by 2 matrices]? [Justify!]

6) [20 points] Let

$$\begin{aligned}
\mathbf{v}_1 &= (-3,-3,-7,-34,-11,3,-33),\\
\mathbf{v}_2 &= (2,2,4,20,6,-2,20),\\
\mathbf{v}_3 &= (1,1,2,10,3,-1,10),\\
\mathbf{v}_4 &= (2,2,5,24,8,-1,21),\\
\mathbf{v}_5 &= (-1,-1,-4,-18,-7,-1,-12),
\end{aligned}$$

and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, and $V = \operatorname{span}(S)$. Given that

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \\ \mathbf{v}_5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 & -1 & 0 & 2 \\ 0 & 0 & 1 & 4 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

answer the questions below. [You do *not* need to justify any of the items below.]

(a) [4 points] What are the dimension of V and $V^{\perp}$ [the orthogonal complement of V in $\mathbb{R}^7$]?

(b) [4 points] Find a basis for V. [No need to be made of elements of S.]

(c) [4 points] Describe $V^\perp$ as a matrix space [row space, column space, null space] of some matrix. [In other words, fill in the blanks of "$V^\perp$ is the ____ space of the matrix ____."]

(d) [4 points] Find the coordinates of each of the vectors in S with respect to the basis you've found in item (b).

(e) [4 points] Which vectors from the standard basis of $\mathbb{R}^7$ can you add to the vectors in the basis of V you've in (b) to obtain a basis of all of $\mathbb{R}^7$?

Scratch: