1) [20 points] *Quickies!* You don't need to justify your answers.

(a) If A is a 2 by 2 matrix with $\det(A) = 3$, then what is $\det(2(A^{\mathrm{T}})^3)$?

Solution. $\det(2(A^{\mathrm{T}})^3) = 2^2 \det((A^{\mathrm{T}})^3) = 2^2 \det(A^{\mathrm{T}})^3 = 2^2 \det(A)^3 = 2^2 \cdot 3^3 = 108.$ □

(b) If $\mathbf{v} = (-2, 1, 2)$ and $\mathbf{w} = (0, -3, 4)$, then what is the *cosine* of the angle between $\mathbf{v}$ and $\mathbf{w}$?

Solution. $\cos(\theta) = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \cdot \|\mathbf{w}\|} = \dfrac{5}{3 \cdot 5} = \dfrac{1}{3}.$ □

(c) If $T : \mathbb{R}^m \to \mathbb{R}^n$ is one-to-one, then what can we say about the sizes of m and n? [In other words, $m < n$, or $m \geq n$, or $m = n$, no restriction, etc.]

Solution. We must have $m \leq n$. □

(d) If $A = \begin{bmatrix} 1 & -2 & 5 & 3 \\ 0 & 2 & 1 & -4 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, then what is the determinant of the matrix obtained by switching two rows of A^{-1}? [If A is not invertible, justify.]

Solution. Since $\det(A) = 12$, we have that $\det(A^{-1}) = 1/12$ and thus the determinant of the matrix in question is $-1/12$. □

(e) Give the two matrices that give the projection onto the xy-plane and the reflection about the yz-plane, respectively.

Solution. They are $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ respectively.

□

2) [20 points] Let $T_1, T_2 : \mathbb{R}^n \to \mathbb{R}^n$ be two *linear* transformations such that T_1 is one-to-one and T_2 is onto.

(a) What can we say about the matrices $[T_1]$ and $[T_2]$?

Solution. Since T_1 is one-to-one we have that $\det([T_1]) \neq 0$ and since T_2 is onto we also have that $\det([T_1]) \neq 0$. [They are also invertible, or have reduced echelon form equal to the identity matrix.] □

(b) Show that the composition $T_2 \circ T_1$ is *both* one-to-one and onto.

Solution. We have $\det([T_2 \circ T_1]) = \det([T_2] \cdot [T_1]) = \det([T_2]) \cdot \det([T_1]) \neq 0$ [as product of non-zero numbers is non-zero]. So, $T_2 \circ T_1$ is both one-to-one and onto. □

3) [20 points]

(a) Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the usual vectors in $\mathbb{R}^2$ and $\mathbf{w} = (\cos(\theta), \sin(\theta))$. Find the projections of $\mathbf{e}_1$ and $\mathbf{e}_2$ on the direction of $\mathbf{w}$. [**Hint:** I shouldn't have to say this, but remember that $\cos^2(\theta) + \sin^2(\theta) = 1$.]

Solution. We have

$$\mathrm{proj}_{\mathbf{w}}(\mathbf{e}_1) = \frac{\cos(\theta)}{(\cos^2(\theta) + \sin^2(\theta))^2}(\cos(\theta), \sin(\theta)) = (\cos^2(\theta), \sin(\theta)\cos(\theta))$$

and

$$\mathrm{proj}_{\mathbf{w}}(\mathbf{e}_2) = \frac{\sin(\theta)}{(\cos^2(\theta) + \sin^2(\theta))^2}(\cos(\theta), \sin(\theta)) = (\sin(\theta)\cos(\theta), \sin^2(\theta)).$$

□

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal *projection* [not reflection!] onto the line on the plane that makes an angle of θ with the x-axis. Find the standard matrix $[T]$ of T. [**Hint:** Item (a) is useful here, as the line in question has the same *direction* as the vector $\mathbf{w}$.]

Solution. Since projection onto the given line is the same as projecting on the direction of $\mathbf{w}$ [i.e., $T(\mathbf{x}) = \mathrm{proj}_{\mathbf{w}}(\mathbf{x})$], we have

$$[T] = \begin{bmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & \sin^2(\theta) \end{bmatrix}.$$

Alternatively, one could find this matrix by composing rotation by $-\theta$, with projection onto the x-axis, with rotation by θ:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & \sin^2(\theta) \end{bmatrix}.$$

□

4) [20 points] Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that the set of all vectors $\mathbf{v} \in \mathbb{R}^n$ such that $T(\mathbf{v}) = \mathbf{0}$ is a vector space.

Solution. We've seen in class that solutions of a homogeneous linear system is a vector space. Now, $T(\mathbf{v}) = \mathbf{0}$ if, and only if, $\mathbf{v}$ is a solution of the homogeneous linear system $[T]\mathbf{x} = \mathbf{0}$. So, the statement is true.

Alternatively, you can do it easily also if you didn't remember the above. Since we are using vectors in $\mathbb{R}^n$ [which is a vector space] with the usual addition and scalar multiplication, we have that we just need to check property 0 from the list. But, if $\mathbf{v}$ and $\mathbf{w}$ are such that $T(\mathbf{v}) = \mathbf{0}$ and $T(\mathbf{w}) = \mathbf{0}$, then [since T is linear] $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w}) = \mathbf{0} + \mathbf{0} = \mathbf{0}$. Also, if $k \in \mathbb{R}$, then [since T is linear] $T(k\,\mathbf{v}) = k\,T(\mathbf{v}) = k\,\mathbf{0} = \mathbf{0}$. So, property 0 holds and the set is a vector space.

□

5) [20 points] Let $V = \mathbb{R}^2$ with the following operations:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \qquad \text{[usual addition]}$$
$$k \cdot (x_1, y_1) = (kx_1, k^2 y_1) \qquad \text{[\textit{unusual} scalar mult.]}$$

Then, V is *not* a vector space. [You can take my word for it.] List **all** items from the list of Vector Space Axioms [given at the end of the test] that fail, and for each item show how it fails by giving a *numerical* example.

Solution. Only item 6 fails: let $\mathbf{u} = (0, 1)$, $k = l = 2$. Then, $(k + l) \cdot \mathbf{u} = 4 \cdot (0, 1) = (0, 16)$, while $(k\,\mathbf{u}) + (l\,\mathbf{u}) = (2 \cdot (0, 1)) + (2 \cdot (0, 1)) = (0, 4) + (0, 4) = (0, 8)$, and hence different. $\square$

Vector Space Axioms

A non-empty set V with a sum and a scalar product is a vector space if it satisfies the following conditions:

0. $\mathbf{u}+\mathbf{v} \in V$ for all $\mathbf{u}, \mathbf{v} \in V$, and $k\mathbf{u} \in V$ for all $\mathbf{u} \in V$ and $k \in \mathbb{R}$;
1. $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$;
2. $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$;
3. there is $\mathbf{0} \in V$ such that $\mathbf{0}+\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$;
4. given $\mathbf{u} \in V$, there exists $-\mathbf{u} \in V$ such that $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$;
5. $k(\mathbf{u}+\mathbf{v}) = k\mathbf{u}+k\mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in V$ and $k \in \mathbb{R}$;
6. $(k+l)\mathbf{u} = k\mathbf{u}+l\mathbf{u}$ for all $\mathbf{u} \in V$ and $k, l \in \mathbb{R}$;
7. $k(l\mathbf{u}) = (kl)\mathbf{u}$ for all $\mathbf{u} \in V$ and $k, l \in \mathbb{R}$;
8. $1\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$.