June 29th, 2010

Math 251

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Summer 2010

Name: ....

Student ID (last 6 digits): XXX- ....

Midterm 2

You have 60 minutes to complete the exam. Do all work on this exam, i.e., on the page of the respective assignment. Indicate clearly, when you continue your solution on the back of the page or another part of the exam.

Write your name and the last six digits of your student ID number on the top of this page. Check that no pages of your exam are missing. This exam has 5 questions and 8 printed pages (including this one, a page with the vector space axioms, and a page for scratch work in the end).

No books, notes or calculators are allowed on this exam!

Show all work! (Unless I say otherwise.) Correct answers without work will receive **zero**. Also, **points will be taken from messy solutions**.

Good luck!

Question	Max. Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1) [20 points] *Quickies!* You don't need to justify your answers.

(a) If A is a 2 by 2 matrix with $\det(A) = 3$, then what is $\det(2(A^{\mathrm{T}})^3)$?

(b) If $\mathbf{v} = (-2, 1, 2)$ and $\mathbf{w} = (0, -3, 4)$, then what is the *cosine* of the angle between $\mathbf{v}$ and $\mathbf{w}$?

(c) If $T : \mathbb{R}^m \to \mathbb{R}^n$ is one-to-one, then what can we say about the sizes of m and n? [In other words, $m < n$, or $m \geq n$, or $m = n$, no restriction, etc.]

(d) If $A = \begin{bmatrix} 1 & -2 & 5 & 3 \\ 0 & 2 & 1 & -4 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, then what is the determinant of the matrix obtained by switching two rows of *the inverse of* A? [If A is not invertible, justify.]

(e) Give the two matrices that give the projection onto the xy-plane and the reflection about the yz-plane, respectively.

2) [20 points] Let $T_1, T_2 : \mathbb{R}^n \to \mathbb{R}^n$ be two *linear* transformations such that T_1 is one-to-one and T_2 is onto.

(a) What can we say about the matrices $[T_1]$ and $[T_2]$?

(b) Show that the composition $T_2 \circ T_1$ is *both* one-to-one and onto.

3) [20 points]

(a) Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the usual vectors in $\mathbb{R}^2$ and $\mathbf{w} = (\cos(\theta), \sin(\theta))$. Find the projections of $\mathbf{e}_1$ and $\mathbf{e}_2$ on the direction of $\mathbf{w}$. [**Hint:** I shouldn't have to say this, but remember that $\cos^2(\theta) + \sin^2(\theta) = 1$.]

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal *projection* [not reflection!] onto the line on the plane that makes an angle of θ with the x-axis. Find the standard matrix $[T]$ of T. [**Hint:** Item (a) is useful here, as the line in question has the same *direction* as the vector $\mathbf{w}$.]

4) [20 points] Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that the set of all vectors $\mathbf{v} \in \mathbb{R}^n$ such that $T(\mathbf{v}) = \mathbf{0}$ is a vector space.

5) [20 points] Let $V = \mathbb{R}^2$ with the following operations:

$$\begin{aligned} (x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2) && \text{[usual addition]} \\ k \cdot (x_1, y_1) &= (kx_1, k^2 y_1) && \text{[\textit{unusual} scalar mult.]} \end{aligned}$$

Then, V is *not* a vector space. [You can take my word for it.] List **all** items from the list of Vector Space Axioms [given at the end of the test] that fail, and for each item show how it fails by giving a *numerical* example.

Vector Space Axioms

A non-empty set V with a sum and a scalar product is a vector space if it satisfies the following conditions:

0. $\mathbf{u} + \mathbf{v} \in V$ for all $\mathbf{u}, \mathbf{v} \in V$, and $k\mathbf{u} \in V$ for all $\mathbf{u} \in V$ and $k \in \mathbb{R}$;
1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$;
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$;
3. there is $\mathbf{0} \in V$ such that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$;
4. given $\mathbf{u} \in V$, there exists $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$;
5. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in V$ and $k \in \mathbb{R}$;
6. $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$ for all $\mathbf{u} \in V$ and $k, l \in \mathbb{R}$;
7. $k(l\mathbf{u}) = (kl)\mathbf{u}$ for all $\mathbf{u} \in V$ and $k, l \in \mathbb{R}$;
8. $1\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$.

Scratch: