1) [20 points] *Quickies!* [These should take you 10 seconds each if you've studied.] You don't need to justify your answers.

(a) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a *one-to-one* linear transformation, then what can we say about the matrix $[T]$?

Solution. $[T]$ is invertible, or $\det([T]) \neq 0$. □

(b) If $T : \mathbb{R}^3 \to \mathbb{R}^2$ is such that $T(\mathbf{e}_1) = (2, 1)$, $T(\mathbf{e}_2) = (1, 0)$, and $T(\mathbf{e}_3) = (-1, 3)$, what is $[T]$?

Solution.

$$[T] = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 3 \end{bmatrix}$$

□

(c) If A is an m by n matrix, then what are the domain and codomain of the linear transformation T_A?

Solution. Domain: $\mathbb{R}^n$. Codomain: $\mathbb{R}^m$. □

(d) If $T : \mathbb{R}^2 \to \mathbb{R}$ is linear with $T(\mathbf{u}) = 2$ and $T(\mathbf{v}) = 3$, then what is $T(\mathbf{u} + 2\mathbf{v})$?

Solution. $T(\mathbf{u} + 2\mathbf{v}) = T(\mathbf{u}) + T(2\mathbf{v}) = T(\mathbf{u}) + 2T(\mathbf{v}) = 8$. □

(e) If the projection of $\mathbf{v}$ on the direction of $\mathbf{u}$ is $(2, 0, 3)$ and the component of $\mathbf{v}$ orthogonal to $\mathbf{u}$ is $(-1, 1, -2)$, then what is $\mathbf{v}$? [This is easier than it might seem at first! *There is very little computation involved!*]

Solution. We have $\mathbf{v} = (2, 0, 3) + (-1, 1, -2) = (1, 1, 1)$. [Draw a picture!] □

2) [15 points] Let $\mathbf{u} = (1, 1, 3)$ and $\mathbf{v} = (0, -2, 0)$.

(a) Find the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$.

Solution.

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} = \frac{-2}{\sqrt{11}\sqrt{4}} = -\frac{\sqrt{11}}{11}.$$

□

(b) Compute $\text{proj}_{\mathbf{u}} \mathbf{v}$.

Solution.

$$\text{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2} \mathbf{u} = \frac{-2}{11} \cdot (1, 1, 3) = \left(-\frac{2}{11}, -\frac{2}{11}, -\frac{6}{11} \right).$$

□

3) [15 points] Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection about the line $y = 2x$.

(a) Find all eigenvalues and the eigenvectors associated to each eigenvalue.

Solution. As with all other reflexions [see the text or notes for reflexions on the axes], we have that the only eigenvalues are 1 and -1, with the eigenvectors associated to 1 exactly the non-zero vectors on the line, i.e., vectors of the form $(a, 2a)$, for $a \in \mathbb{R} - \{0\}$, and eigenvectors associated to -1 exactly the non-zero vectors perpendicular to the line, i.e., vectors of the form $(-2a, a)$, for $a \in \mathbb{R} - \{0\}$.

[Think about it geometrically! Again, this is exactly like the example of reflexions about the coordinate axes done in class and in the book.]

$\square$

(b) Is the matrix $[T]$ invertible? [Justify!]

Solution. We have that T is invertible, as if you reflect about the same line twice, we get the original vector back. [I.e., T is its own inverse, or, in symbols, $T = T^{-1}$.] Hence, $[T]$ is an invertible matrix. [In fact $[T]^1 = [T^{-1}] = [T]$, so the inverse of $[T]$ is also $[T]$.]

$\square$

4) [15 points] Let $S = \{(2,3,6),(4,1,7),(6,2,11)\}$. This set is not linearly independent and does not span $\mathbb{R}^3$. [Just take my word for it!] Find a linear combination of the vectors of S that give the zero vector *with at least one coefficient being non-zero* and a vector not in $\operatorname{span}(S)$.
[**Hint:** You've done both in HW. *You can solve both questions together here!*]

Solution.

$$\left[\begin{array}{ccc|c|c} 2 & 4 & 6 & 0 & a \\ 3 & 1 & 2 & 0 & b \\ 6 & 7 & 11 & 0 & c \end{array}\right] \sim \left[\begin{array}{ccc|c|c} 1 & 2 & 3 & 0 & a/2 \\ 3 & 1 & 2 & 0 & b \\ 6 & 7 & 11 & 0 & c \end{array}\right] \sim$$
$$\left[\begin{array}{ccc|c|c} 1 & 2 & 3 & 0 & a/2 \\ 0 & -5 & -7 & 0 & b-3a/2 \\ 0 & -5 & -7 & 0 & c-3a \end{array}\right] \sim \left[\begin{array}{ccc|c|c} 1 & 2 & 3 & 0 & a/2 \\ 0 & -5 & -7 & 0 & b-3a/2 \\ 0 & 0 & 0 & 0 & c-b-3a/2 \end{array}\right]$$

This gives us that, for instance $(1,0,0)$ is not in the range, as $0-0-3/2 \neq 0$. So, we continue only with the homogeneous part now:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 7/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1/5 & 0 \\ 0 & 1 & 7/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So, we have a solution $x_1 = -t/5$, $x_2 = -7t/5$, and $x_3 = t$. To find a non-zero solution, we just take some $t \neq 0$, say $t = 5$. Then, we have:

$$-(2,3,6) - 7(4,1,7) + 5(6,2,11) = (0,0,0).$$

□

5) [15 points] Is the set of all 2 by 2 matrices of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \qquad \text{with } a, b \in \mathbb{R},$$

a vector space with the usual addition and scalar multiplication of matrices? [Justify!]

Solution. Yes! We have that this set is a subset of the vector space $M_{2\times 2}$ [or M_{22}, as the book writes], and hence it suffices to check it is a subspace. Let's call the given set V. [Note that V is just the set of all 2 by 2 diagonal matrices.]

We clearly have that the zero matrix is in V, as we can take $a = b = 0$.

We have that

$$\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix} + \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & 0 \\ 0 & b_1 + b_2 \end{bmatrix} \in V$$

[as it is a diagonal matrix].

Finally we also have

$$k \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} ka & 0 \\ 0 & kb \end{bmatrix} \in V$$

[as it is a diagonal matrix]. □

6) [20 points] Let $V = (0, +\infty)$ [i.e., all positive real numbers] with the following operations:

$$x \oplus y = xy, \qquad k \odot x = x^k \qquad [x, y \in V,\ k \in \mathbb{R}]$$

[Just like in your HW! I am using "$\oplus$" and "$\odot$" to denote the sum and scalar multiplication to avoid confusion with the regular operations, though.] The set V with these operations *is* a vector space. [Take my word for it.]

(a) Check that $k \odot (x \oplus y) = (k \odot x) \oplus (k \odot y)$.

Solution. We have

$$\begin{aligned} k \odot (x \oplus y) &= k \odot (xy) \\ &= (xy)^k \\ &= x^k y^k \\ &= (k \odot x)(k \odot y) \\ &= (k \odot x) \oplus (k \odot y) \end{aligned}$$

□

(b) What is the "zero" of this vector space [i.e., the element $\theta \in V$ such that $\theta \oplus x = x$ for all $x \in V$]?

Solution. It is 1, as $1 \oplus x = x^1 = x$ for all $x \in V$. □

(c) Given $x \in V$, what is its "negative" of x [i.e., the element $y \in V$ such that $x \oplus y = \theta$, where θ is the zero from the previous item]?

Solution. It is x^{-1}. [Note that since $V = (0, +\infty)$, we have that $x^{-1} = 1/x$ is always valid, as we will not get a zero in the denominator.] Then, $x \oplus x^{-1} = xx^{-1} = 1$. □