Final

Math 351 – Spring 2020

May 1st, 2020

Instructions

- *Write neatly and legibly!*
- Your camera *must* be on at *all times* and showing you properly. (You can only leave Zoom when you are done!)
- Leave the sound on (not the mic), so that you can *hear* incoming private messages or if I need to say something to all.
- You do not need to copy the statements. Just number your answers.
- Each problem must be solved in a different page, but items of the same problem can be in the same page.
- If you have any questions, send me a private message through the chat.
- You can only use your computer to look at the exam or to send me a message.
- **When you are done with the exam and are about to start scanning/uploading, send me a private message!** (Something like *"Scanning now."*)
- Make sure your scans are legible before uploading them to Canvas.
- **When you are done uploading your solutions, send me a private message.** (Something like *"Done."* No need for the time.) You can then leave Zoom.
- **Be prepared to, upon request (via private message), show me your surroundings!**

1) [10 points] Find the remainder of 2^{2020} when divided by 7. [**No calculators!** Show your computations!]

2) [10 points] Find all $x \in \mathbb{Z}$ satisfying [simultaneously]:

$$\begin{aligned} x &\equiv 3 \pmod{5}, \\ 2x &\equiv 3 \pmod{7}. \end{aligned}$$

If there is no such x, simply justify why.

3) [10 points] Prove that if $\gcd(r, m) = \gcd(r', m) = 1$, then $\gcd(rr', m) = 1$.

[**Note:** This was a HW problem.]

4) [10 points] Let F be a field and $f(x) \in F[x]$. Prove that if $f(x^2)$ is irreducible, then so is $f(x)$.

5) Suppose that F is a field, $a \in F$ and $f \in F[x]$. Prove that if $(x - a) \mid f$ and $(x - a) \mid f'$ [where f' is the *derivative* of f], then $(x - a)^2 \mid f$.

[**Note:** This was a HW problem. **Hint:** You can use the *Basic Lemma* for polynomials: Assume that $f \mid g$. Then, $f \mid (g + h)$ iff $f \mid h$.]

6) Examples:

(a) [5 points] Give an example of an *infinite* field F such that for all $a \in F$, we have $2020 \cdot a = 0$.

(b) [5 points] Give an example of an *infinite* commutative ring which is *not* a domain.

Continues on next page!

7) Determine if the polynomials below are irreducible or not in the corresponding polynomial ring. *Justify each answer!*

(a) [3 points] $f = x^2 - 2x + 3$ in $\mathbb{R}[x]$.

(b) [3 points] $f = x^{2020} - 2020$ in $\mathbb{C}[x]$.

(c) [3 points] $f = 137x + 389$ in $\mathbb{F}_{521}[x]$.

(d) [3 points] $f = x^5 + 400x^4 - 10x^3 + 120x^2 - 3000x + 310$ in $\mathbb{Q}[x]$.

(e) [4 points] $f = x^3 + 2x^2 - 2x + 1$ in $\mathbb{Q}[x]$.

(f) [4 points] $f = 30003x^3 - 10x^2 + 11x + 30001$ in $\mathbb{Q}[x]$.

8) Let $\sigma, \tau \in S_9$ be given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 5 & 6 & 2 & 4 & 8 & 1 & 9 & 3 \end{pmatrix} \quad \text{and} \quad \tau = (1\ 3\ 8\ 2)(4\ 5\ 9).$$

(a) [3 points] Write the *complete* factorization of σ into disjoint cycles.

(b) [3 points] Compute τ^{-1}. [Your answer can be in any form.]

(c) [3 points] Compute $\tau\sigma$. [Your answer can be in any form.]

(d) [3 points] Compute $\sigma\tau\sigma^{-1}$. [Your answer can be in any form.]

(e) [3 points] Write τ as a product of transpositions.

(f) [2 points] Compute $\mathrm{sign}(\tau)$.

(g) [3 points] Compute $|\tau|$ (the order of τ in S_9).