Midterm 4

Math 351 – Spring 2020

April 15th, 2020

1) [25 points] Let $f = x^5 + x^4 + x^2 + 2x + 1$ and $g = x^4 + 2x$ in $\mathbb{F}_3[x]$. Find $\gcd(f, g)$ [in $\mathbb{F}_3[x]$]. [No need to express the GCD as a linear combination.]

Solution. We have:

$$f = g \cdot (x + 1) + (2x^2 + 1),$$

$$g = (2x^2 + 1) \cdot (2x^2 + 2) + (2x + 1),$$

$$(2x^2 + 1) = (2x + 1) \cdot (x + 1) + 0.$$

So, the GCD is the "monic version" of $2x + 1$, i.e., $2(2x + 1) = x + 2$.

□

2) [25 points] Let R be a domain. Prove that $R[x]$ is never a field.
[This was a HW problem.]

Proof. We know that $R[x]$ is a domain, so we need to show that there is some non-zero, non-invertible element. We will show that x is such an element.
Suppose $x \cdot f = 1$, for some $f \in R[x]$. Clearly $f \neq 0$, or else $x \cdot f = x \cdot 0 = 0 \neq 1$. Hence, $\deg(f) \geq 0$. Then, since R is a domain, $0 = \deg(1) = \deg(x \cdot f) = \deg(x) + \deg(f) = 1 + \deg(f) \geq 1$, a contradiction. □

3) [25 points] Let F be a field and $f, g \in F[x]$, with $\deg(f) = \deg(g) > 0$, both *monic*. Prove that if $f \mid g$, then $f = g$.

Proof. Suppose $f \mid g$, i.e., $g = f \cdot h$ for some $h \in F[x]$. Then, since F is a domain, we have $\deg(f) = \deg(g) = \deg(f \cdot h) = \deg(f) + \deg(h)$, and so $\deg(h) = 0$, i.e., $h \in F^{\times}$, say $h = a \in F^{\times}$.
Now, since f is monic, the leading coefficient of $a \cdot f$ is a [as $a \cdot f = a \cdot (x^n + \cdots) = ax^n + \cdots$]. But $a \cdot f = g$ and g is also monic, so $a = 1$ and $g = f$. □

4) [25 points] Let F be a field, $f \in F[x]$ and assume f is irreducible. Prove that for any $a \in F^{\times}$, the polynomial $a \cdot f$ is also irreducible.

Proof. Suppose that $a \cdot f$ is *reducible*, i.e., that $a \cdot f = g \cdot h$, for some $g, h \in F[x]$ with $\deg(g), \deg(h) \geq 1$. Then, $f = a^{-1} \cdot (a \cdot f) = a^{-1} \cdot (g \cdot h) = (a^{-1} \cdot g) \cdot h$. Since F is a domain, $\deg(a^{-1} \cdot g) = \deg(a^{-1}) + \deg(g) = 0 + \deg(g) = \deg(g) \geq 1$. Hence f is also reducible. □