1) Give the *set of units* [i.e., invertible elements] for the rings below. [These don't need to be justified, but if there are no justifications, I cannot give partial credit.]

(a) [5 points] $\mathbb{Z}$

(b) [5 points] $\mathbb{R}$

(c) [5 points] $\mathbb{F}_{13}$

(d) [10 points] $\mathbb{I}_{15}$ [same as $\mathbb{Z}/15\mathbb{Z}$].

Solution. (a) $\{\pm 1\}$

(b) $\mathbb{R} \setminus \{0\}$

(c) $\mathbb{F}_{13} \setminus \{0\}$

(d) $\mathbb{I}_{15} = \{1, 2, 4, 7, 8, 11, 13, 14\}$.

□

2) [25 points] Prove that the prime field of $\mathbb{R}$ is $\mathbb{Q}$.
[**Hint:** This was a HW problem.]

Proof. Let F be the prime field of $\mathbb{R}$. Since $\mathbb{Q}$ is a subfield of $\mathbb{R}$, by the minimality of the field of fractions, $F \subseteq \mathbb{Q}$.
Now, since F is a subring of $\mathbb{R}$, we have that $1 \in F$, and hence, since it is a ring, we have that $\mathbb{Z} \subseteq F$ [as F is closed under sums and subtractions]. But this implies that the field of fractions of $\mathbb{Z}$, namely $\mathbb{Q}$, is contained in F [by the minimality of the field of fractions].
With the two inclusions we get $F = \mathbb{Q}$. □

3) [25 points] Prove that

$$\mathbb{Z}[\sqrt[3]{2}] \stackrel{\text{def}}{=} \{a + b\sqrt[3]{2} + c\sqrt[3]{4} \ : \ a, b, c \in \mathbb{Z}\}$$

is a domain.
[**Hint:** There is a hard and an easy way to do this. Part of this was done in class.]

Proof. It suffices to show that $\mathbb{Z}[\sqrt[3]{2}]$ is a subring of $\mathbb{R}$.
First,

$$1 = 1 + 0 \cdot \sqrt[3]{2} + 0 \cdot \sqrt[3]{4} \in \mathbb{Z}[\sqrt[3]{2}].$$

Also, let $\alpha, \beta \in \mathbb{Z}[\sqrt[3]{2}]$. Then, $\alpha = a_1 + b_1\sqrt[3]{2} + c_1\sqrt[3]{4}$, $\beta = a_2 + b_2\sqrt[3]{2} + c_2\sqrt[3]{4}$ for some $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{Z}$. Then:

$$\alpha - \beta = (a_1 + b_1\sqrt[3]{2} + c_1\sqrt[3]{4}) - (a_2 + b_2\sqrt[3]{2} + c_2\sqrt[3]{4}) = (a_1 - a_2) + (b_1 - b_2) \cdot \sqrt[3]{2} + (c_1 - c_2)\sqrt[3]{4}.$$

Since $a_1 - a_2, b_1 - b_2, c_1 - c_2 \in \mathbb{Z}$ [since $\mathbb{Z}$ is closed under subtractions], we have that $\alpha - \beta \in \mathbb{Z}[\sqrt[3]{2}]$.
Finally,

$$\begin{aligned} \alpha \cdot \beta &= (a_1 + b_1\sqrt[3]{2} + c_1\sqrt[3]{4}) \cdot (a_2 + b_2\sqrt[3]{2} + c_2\sqrt[3]{4}) \\ &= (a_1a_2 + 2b_1c_2 + 2c_1b_2) + (a_1b_2 + b_1a_2 + 2c_1c_2)\sqrt[3]{2} + (a_1c_2 + b_1b_2 + c_1a_2)\sqrt[3]{4}. \end{aligned}$$

Since $\mathbb{Z}$ is closed under sums and products, we have that $\alpha \cdot \beta \in \mathbb{Z}[\sqrt[3]{2}]$.
The three steps above show that $\mathbb{Z}[\sqrt[3]{2}]$ is a subring of $\mathbb{R}$. Since $\mathbb{R}$ is a field, every subring of $\mathbb{R}$ is a domain. Hence, $\mathbb{Z}[\sqrt[3]{2}]$ is a domain. □

4) [25 points] Prove, using only the axioms for commutative rings [listed on the last page], that if R is a commutative ring, then for all $a \in R$ we have that $a \cdot 0 = 0$. *You have to justify every step of your proof!*

[**Hints:** This was done in class! Use Axiom 3 to write $0 = 0 + 0$. Be careful to use the associative [Axiom 2] and commutative [Axiom 1] when necessary!]

Proof. By Axiom 3, we have that $0 = 0 + 0$. Then:

$$a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0, \tag{1}$$

using Axiom 8. Then:

$$\begin{aligned} 0 &= a \cdot 0 + (-(a \cdot 0)) && \text{[by Axiom 4]} \\ &= (a \cdot 0 + a \cdot 0) + (-(a \cdot 0)) && \text{[by Eq. (1)]} \\ &= a \cdot 0 + (a \cdot 0 + (-(a \cdot 0))) && \text{[by Axiom 2]} \\ &= a \cdot 0 + 0 && \text{[by Axiom 4]} \\ &= a \cdot 0 && \text{[by Axiom 3].} \end{aligned}$$

□

Commutative Ring Axioms: A [non-empty] set with two operations, $+$ and $\cdot$, is a commutative ring if:

0. For all $a, b \in R$ we have that $a + b \in R$ and $a \cdot b \in R$.
1. For all $a, b \in R$ we have that $a + b = b + a$.
2. For all $a, b, c \in R$ we have that $(a + b) + c = a + (b + c)$.
3. There exists $0 \in R$ such that for all $a \in R$ we have $a + 0 = a$.
4. For all $a \in R$ there exists $-a \in R$ such that $a + (-a) = 0$.
5. For all $a, b \in R$ we have that $a \cdot b = b \cdot a$.
6. For all $a, b, c \in R$ we have that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
7. There is $1 \in R$ such that for all $a \in R$ we have that $1 \cdot a = a$
8. For all $a, b, c \in R$ we have that $a \cdot (b + c) = a \cdot b + a \cdot c$