Midterm 3

Math 351 – Spring 2020

March 25th, 2020

Instructions

- *Write neatly and legibly!*
- Your camera *must* be on at *all times* and showing you properly. (You can only leave Zoom when you are done!)
- Leave the sound on, so that you can hear incoming private messages or if I need to say something to all.
- You do not need to copy the statements. Just number your answers.
- Each problem must be solved in a different page, but items of the same problem can be in the same page.
- If you have any questions, send me a private message through the chat.
- You can only use your computer to look at the exam or to send me a message.
- When you are done, scan your solutions, make sure they are legible, and upload them to Canvas.
- **When you are done uploading your solutions, send me a private message with the time.** Then you can leave Zoom.
- **Be prepared to, upon request (via private message), show me your surroundings!**

1) Give the *set of units* [i.e., invertible elements] for the rings below. [These don't need to be justified, but if there are no justifications, I cannot give partial credit.]

(a) [5 points] $\mathbb{Z}$

(b) [5 points] $\mathbb{R}$

(c) [5 points] $\mathbb{F}_{13}$

(d) [10 points] $\mathbb{I}_{15}$ [same as $\mathbb{Z}/15\mathbb{Z}$].

2) [25 points] Prove that the prime field of $\mathbb{R}$ is $\mathbb{Q}$.
[**Hint:** This was a HW problem.]

3) [25 points] Prove that

$$\mathbb{Z}[\sqrt[3]{2}] \stackrel{\text{def}}{=} \{a + b\sqrt[3]{2} + c\sqrt[3]{4} \ : \ a, b, c \in \mathbb{Z}\}$$

is a domain.
[**Hint:** There is a hard and an easy way to do this. Part of this was done in class.]

4) [25 points] Prove, using only the axioms for commutative rings [listed on the last page], that if R is a commutative ring, then for all $a \in R$ we have that $a \cdot 0 = 0$. *You have to justify every step of your proof!*
[**Hints:** This was done in class! Use Axiom 3 to write $0 = 0 + 0$. Be careful to use the associative [Axiom 2] and commutative [Axiom 1] when necessary!]

Commutative Ring Axioms: A [non-empty] set with two operations, $+$ and $\cdot$, is a commutative ring if:

0. For all $a, b \in R$ we have that $a + b \in R$ and $a \cdot b \in R$.
1. For all $a, b \in R$ we have that $a + b = b + a$.
2. For all $a, b, c \in R$ we have that $(a + b) + c = a + (b + c)$.
3. There exists $0 \in R$ such that for all $a \in R$ we have $a + 0 = a$.
4. For all $a \in R$ there exists $-a \in R$ such that $a + (-a) = 0$.
5. For all $a, b \in R$ we have that $a \cdot b = b \cdot a$.
6. For all $a, b, c \in R$ we have that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
7. There is $1 \in R$ such that for all $a \in R$ we have that $1 \cdot a = a$
8. For all $a, b, c \in R$ we have that $a \cdot (b + c) = a \cdot b + a \cdot c$