Defining Automorphisms

Math 552

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One cannot simply definite automorphims (e.g., elements of the Galois group) by simply choosing [even if carefully] the elements to which they are sent.

Example: Let$F \stackrel{\text{def}}{=} \mathbb{Q}$ and K be the splitting field of $f \stackrel{\text{def}}{=} x^8 - 2$. Then $K = F[\sqrt[8]{2}, \zeta]$, where $\zeta = \zeta_8 = e^{2\pi i/8}$. Let then $\sigma \in \mathrm{Gal}(K/F)$ such that $\sigma(\sqrt[8]{2}) = \sqrt[8]{2} \cdot \zeta$ (another root of the irreducible $f = m_{\sqrt[8]{2},F}$) and $\sigma(\zeta) = \zeta$.

This is *incredibly common* to see done, but you should not do it! This σ does not exist!

If it did: since $\sigma(\sqrt[8]{2}) = \sqrt[8]{2} \cdot \zeta$, we have that $\sigma(\sqrt{2}) = \sigma((\sqrt[8]{2})^4) = (\sqrt[8]{2} \cdot \zeta)^4 = \sqrt{2} \cdot \zeta^4 = -\sqrt{2}$. Also, since $\zeta^2 = i$ and $\sigma(\zeta) = \zeta$, we have that $\sigma(i) = \sigma(\zeta^2) = \zeta^2 = i$.

Now note that $\zeta = \sqrt{2}/2 + i\sqrt{2}/2$. Now, from above, $\sigma(\sqrt{2}) = -\sqrt{2}$, $\sigma(i) = i$, so $\sigma(\zeta) = \sigma(\sqrt{2}/2 + i\sqrt{2}/2) = \sigma(\sqrt{2})/2 + \sigma(i)\sigma(\sqrt{2})/2 = -\sqrt{2}/2 - i\sqrt{2} = -\zeta$. But that is a contradiction, since we had defined $\sigma(\zeta) = \zeta$. $\square$

I hope this clarifies the matter, and you will avoid defining automorphisms like above in the future!