Integration Review

1 Basic Functions

1. $\int k\,dx = kx + C$ for any constant k
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ except for $n = -1$.
3. $\int \frac{1}{x}\,dx = \ln|x| + C$
4. $\int e^x\,dx = e^x$
5. $\int \cos(x)\,dx = \sin(x)$
6. $\int \sin(x)\,dx = -\cos(x)$

Properties of Antiderivatives:

1. $\int kf(x)\,dx = k(\int f(x)\,dx)$
2. $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

If you want to find the antiderivative of a function that is NOT one of these basic functions, you need to use an integration technique!

Primary Techniques:

1. Substitution
2. Integration by Parts
3. Simplify integrand using trig identities
4. Trig. Substitution
5. Partial Fractions

2 Substitution

This is generally the technique I try first, partly because it tends to work more often that not, and partly just because I prefer it to the others..

This technique **undoes** the chain rule! Remember if you have $f(g(x))$, the composition of two functions, then

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

Integrating both sides, I get:

$$f(g(x)) + C = \int f'(g(x))g'(x)\,dx$$

The issue, of course, is that it's not always so clear given an integration problem whether or not the integrand can be seen as the result of the chain rule! Basically subsitution allows us to see this:

Let $u = g(x)$, then $du = g'(x)\,dx$ and

$$f(u) + C = \int f'(u)\,du$$

The point here being that if the integrand really is the result of a chain rule, substitution will usually change the form of the problem into something that we can integrate directly!

Example: Integrate $\int 6x(3x^2+5)^3\,dx$.

Solution: Let $u = 3x^2 + 5$. Then $du = 6x\,dx$ which appears exactly in the problem. We can change variables, then, to get

$$\int 6x(3x^2+5)^3\,dx = \int u^3\,du = \frac{u^4}{4} + C = \frac{(3x^2+5)^4}{4} + C$$

Notice that once we made the substitution, we reduced the problem down to one of integrating a basic function!

Typical Problems

1. $\int \frac{2}{3-x}\,dx$

   Let $u = 3 - x$, then $du = -dx$, and

$$\int \frac{2}{3-x}\,dx = \int \frac{-2}{u}\,du = -2\ln|u| + C = -2\ln|3-x| + C$$

2. $\int \cos(2x+1)\,dx$

   Let $u = 2x + 1$, then $du = 2dx$, so $dx = du/2$ and

$$\int \cos(2x+1)\,dx = \frac{1}{2}\int \cos(u)\,du = \frac{1}{2}\sin(u) + C = \frac{1}{2}\sin(2x+1) + C$$

3. $\int 3xe^{2x^2+4}\,dx$

   Let $u = 2x^2 + 4$, then $du = 4x\,dx$, so $xdx = du/4$ and

$$\int 3xe^{2x^2+4}\,dx = \int \frac{3}{4}e^u\,du = \frac{3}{4}e^u + C = \frac{3}{4}e^{2x^2+4} + C$$

4. $\int x\sqrt{2x+5}\,dx$

   Let $u = 2x + 5$, then $du = 2dx$, so $dx = du/2$, $x = \frac{u-5}{2}$, and

$$\int x\sqrt{2x+5}\,dx = \frac{1}{2}\int \frac{u-5}{2}\sqrt{u}\,du = \frac{1}{4}\int (u-5)u^{1/2}\,du = \frac{1}{4}\int u^{3/2} - 5u^{1/2}\,du$$

$$= \frac{1}{10}u^{5/2} - \frac{5}{6}u^{3/2} + C = \frac{1}{10}(2x+5)^{5/2} - \frac{5}{6}(2x+5)^{3/2} + C$$

5. $\int \frac{2+x}{3-x}\,dx$

   Let $u = 3 - x$, then $du = -dx$ and $x = 3 - u$, so

$$\int \frac{2+x}{3-x}\,dx = -\int \frac{5-u}{u}\,du = \int 1 - \frac{5}{u}\,du = u - 5\ln|u| + C = (3-x) - 5\ln|3-x| + C$$

3 Integration by Parts

This technique is designed to undo the product rule. Remember

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

so integrating both sides, we get

$$f(x)g(x) + C = \int (f'(x)g(x) + f(x)g'(x))\, dx$$

and if we rearrange a little bit, we have

$$\int f(x)g'(x)\, dx = fg - \int f'g\, dx \ .$$

The notation we usually is: let $u = f$ and $v = g$, then $du = f'(x)\, dx$ and $dv = g'(x)\, dx$. Substituting, we get

$$\int udv = uv - \int vdu \ .$$

This is generally used when you have a product of two function in the integrand and substitution does not work.

Example:

$$\int xe^x\, dx$$

Solution: Let $u = x$, and $dv = e^x\, dx$. Then $du = dx$ and $v = e^x$, and we can get

$$\int xe^x\, dx = xe^x - \int e^x\, dx = xe^x - e^x + C \ .$$

Typical Problems

1. $\int x \ln x \, dx$

   Let $u = \ln x$, and $dv = x\,dx$ (notice that the other choice is more difficult, because if $dv = \ln x\,dx$, we cannot find v easily). Now, $du = \frac{1}{x}\,dx$ and $v = x^2/2$.

$$\int x \ln x \, dx = \frac{x^2 \ln x}{2} - \int \frac{x}{2}\,dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$$

2. $\int x^2 \cos x \, dx$

   Let $u = x^2$ and $dv = \cos x\,dx$, then $du = 2x\,dx$ and $v = \sin x$, so

$$\int x^2 \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx$$

   Now, let $u = 2x$ and $dv = \sin x\,dx$, then $du = 2\,dx$ and $v = -\cos x$, so

$$= x^2 \sin x - [-2x \cos x + \int 2 \cos x \, dx] = x^2 \sin x + 2x \cos x - 2 \sin x + C$$

3. $\int e^x \cos x \, dx$

   Let $u = e^x$ and $dv = \cos x\,dx$, then $du = e^x\,dx$ and $v = \sin x$, so

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

   let $u = e^x$ and $dv = \sin x\,dx$, then $du = e^x\,dx$ and $v = -\cos x\,dx$, so

$$= e^x \sin x - [-e^x \cos x + \int e^x \cos x \, dx]$$

   putting everything together, we see that

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

   Adding $\int e^x \cos x \, dx$ to both sides, we get

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x$$

   so that finally,

$$\int e^x \cos x \, dx = \frac{1}{2}[e^x \sin x + e^x \cos x]$$

4. $\int (x+3) \sin(x+2)\, dx$

   Let $w = x + 2$, so $dw = dx$ and $x = w - 2$ (note you don't have to do this part, I'm just trying to "clean things up" a bit before doing the integration by parts)

$$\int (x+3)\sin(x+2)\,dx = \int (w+1) \sin w \, dw = \int w \sin w \, dw + \int \sin w \, dw = \int w \sin w \, dw - \cos w + C$$

   Let $u = w$ and $dv = \sin w\,dw$, then $du = dw$ and $v = -\cos w$, so

$$= -w \cos w + \int \cos w \, dw - \cos w + C = -w \cos w + \sin w - \cos w + C$$

   Subbing back in for $w = x + 2$,

$$= -(x+3)\cos(x+2) + \sin(x+2) + C$$

5. $\int \ln x \, dx$

This one is definately not an obvious integration by parts, but it does work! Let $u = \ln x$ and $dv = 1\, dx$, then $du = \frac{1}{x}\, dx$ and $v = x$, so

$$\int \ln x \, dx = x \ln x - \int dx = x \ln x - x + C$$

4 Common Integrals involving Trig Functions - Using Trig Identities to Simplify Integrals

In general, you'll use one of the following (or some rearrangement of one of these...)

A. $1 = \cos^2\theta + \sin^2\theta$

B. $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$

C. $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$

Typical Examples

1. $\int \cos\theta \sin\theta \, d\theta$ (solve by substitution)

   Let $u = \sin\theta$, then $du = \cos\theta \, d\theta$ and

$$\int \cos\theta \sin\theta \, d\theta = \int u \, du = u^2/2 + C = \frac{\sin^2\theta}{2} + C$$

2. $\int \cos^2\theta \, d\theta$ (use trig identity B.)

$$\int \cos^2\theta \, d\theta = \int \frac{1+\cos(2\theta)}{2} \, d\theta = \frac{\theta}{2} + \frac{\sin(2\theta)}{4} + C$$

3. $\int \sin^2\theta \, d\theta$ (use trig identity C.)

$$\int \sin^2\theta \, d\theta = \int \frac{1-\cos(2\theta)}{2} \, d\theta = \frac{\theta}{2} - \frac{\sin(2\theta)}{4} + C$$

4. $\int \cos^3\theta \, d\theta$ (start by changing $\cos^2\theta = 1 - \sin^2\theta$, then use substitution)

$$\int \cos^3\theta \, d\theta = \int (1 - \sin^2\theta) \cos\theta \, d\theta = \int \cos\theta \, d\theta - \int \sin^2\theta \cos\theta \, d\theta = \sin\theta - u^3/3 + C$$

   where we have let $u = \sin\theta$, so $du = \cos\theta \, d\theta$. Finally

$$= \sin\theta - \sin^3\theta/3 + C$$

5 Trig Substitution

This is based on using a trig identity to simplify the form of a problem, but here we begin with a problem that does not involve trig functions. The basic indentities used are usually

$$\cos^2\theta + \sin^2\theta = 1$$

and

$$1 + \tan^2\theta = \sec^2\theta$$

Example $\int \sqrt{1-x^2}\,dx$

Solution Since $\cos^2\theta + \sin^2\theta = 1$, this implies that $1 - \sin^2\theta = \cos^2\theta$. So, if we let $x = \sin\theta$, which implies that $dx = \cos\theta\, d\theta$, we get

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2\theta}\cos\theta\, d\theta = \int \cos^2\theta\, d\theta$$

So, by changing the integral to be in terms of our trig substitution, we were able to eliminate the square root... we still aren't quite done, but it's a start!

We could finish by using the trig identity $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$:

$$= \int \frac{1+\cos(2\theta)}{2}\, d\theta = \int 1/2\, d\theta + 1/2 \int \cos(2\theta)\, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C$$

Finally, we need to put this back in terms of x, so:

$$= \frac{1}{2}\arcsin(x) + \frac{1}{2}\cos\theta\sin\theta + C = \frac{1}{2}\arcsin(x) + \frac{x}{2}\sqrt{1-x^2} + C$$

Example $\int \sqrt{4-x^2}\,dx$ (let $x = 2\sin\theta$)

$$\int \sqrt{4-x^2}\,dx = \int \sqrt{4-4\sin^2\theta}\,2\cos\theta\,d\theta = \int 4\cos^2\theta\,d\theta = 2\theta + \sin(2\theta) + C$$

where we get the last integral because we already found $\int \cos^2\theta\,d\theta$ above. Now, since $\theta = \arcsin(x/2)$, we have

$$= 2\arcsin(x/2) + \sin(2\arcsin(x/2)) + C$$

Example $\int \frac{1}{1+x^2}\,dx$ (let $x = \tan\theta$, so $dx = \sec^2\theta\,d\theta$)

$$\int \frac{1}{1+x^2}\,dx = \int \frac{sec^2\theta}{1+tan^2\theta}\,d\theta = \int d\theta = \theta + C = \arctan(x) + C$$

6 Partial Fractions

This is probably the least commonly used, but does appear from time to time... Basically, we find a way to rewrite the quotient of two polynomials in a way that makes integration easier.

Example $\int \frac{1}{x^2-4x+3}\,dx$ **Solution** First of all, notice that $\frac{1}{x^2-4x+3} = \frac{1}{(x-3)(x-1)}$. We now try to find A and B, both constants, such that

$$\frac{1}{(x-3)(x-1)} = \frac{A}{x-3} + \frac{B}{x-1}$$

To find A and B, multiply both sides by $(x-3)(x-1)$ to get

$$1 = A(x-1) + B(x-3) = (A+B)x + (-A-3B)$$

where we collected like terms in the end. If you compare the polynomials on each side, you see that the constant term should be 1 and there should be no x term. Thus

$$A + B = 0$$

$$-A - 3B = 1$$

Solving this system gives:

$$-2B = 1$$

or $B = -1/2$. Since $A = -B$, $A = 1/2$. We end up with

$$\frac{1}{(x-3)(x-1)} = \frac{1/2}{x-3} + \frac{-1/2}{x-1}$$

so that

$$\int \frac{1}{x^2-4x+3}\,dx = \int \frac{1/2}{x-3} - \frac{1/2}{x-1}\,dx = \frac{1}{2}\ln|x-3| - \frac{1}{2}\ln|x-1| + C$$

where you can use substitution to get the final integrals.

Example For something like

$$\int \frac{1}{(x^2+1)(x-2)}\,dx$$

since the factor x^2+1 doesn't factor further, the P.F.D. looks like

$$\frac{1}{(x^2+1)(x-2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-2}$$

(the polynomial on top in the decomposition should always be one degree less than the one on the bottom, unless it's a repeated factor) This implies that

$$1 = (Ax+B)(x-2) + C(x^2+1) = (A+C)x^2 + (-2A+B)x + (-2B+C)$$

so that

$$1 = -2B + C$$

$$0 = A + C$$

$$0 = -2A + B$$

Solving this system we get $A = -1/5$, $B = -2/5$, and $C = 1/5$, so

$$\int \frac{1}{(x^2+1)(x-2)}\,dx = \frac{1}{5}\left(\int \frac{-x-2}{x^2+1} + \frac{1}{x-2}\,dx\right) = \frac{1}{5}\left[-\int \frac{x}{x^2+1}\,dx - \int \frac{2}{x^2+1}\,dx + \ln|x-2| + C\right]$$

For the first integral, we can use substitution letting $u = x^2 + 1$, but the second integral requires trig substitution! The final integral is

$$= -\frac{1}{10}\ln(x^2+1) + \frac{1}{5}\ln|x-2| - \frac{2}{5}\arctan(x) + C$$

7 A Difficult Example that requires trig sub. AND partial fractions

Example

$$\int \sqrt{1+x^2}\,dx$$

Let $x = \tan\theta$, then $dx = \sec^2\theta\,d\theta$ and

$$\int \sqrt{1+x^2}\,dx = \int \sec^3\theta\,d\theta = \int \frac{1}{\cos^3\theta}\,d\theta = \int \frac{\cos\theta}{\cos^4\theta}\,d\theta$$

$$= \int \frac{\cos\theta}{(1-\sin^2\theta)^2}\,d\theta = \int \frac{1}{(1-u^2)^2}\,du = \int \frac{1/4}{1-u} + \frac{1/4}{(1-u)^2} + \frac{1/4}{1+u} + \frac{1/4}{(1+u)^2}\,du$$

$$= -\frac{1}{4}\ln(1-u) + \frac{1}{4}\ln(1+u) + \frac{1}{4(1-u)} - \frac{1}{4(1+u)} + C = \frac{1}{4}\ln(\frac{1+u}{1-u}) + \frac{u}{2(1-u^2)} = \frac{1}{4}\ln(\frac{1+\sin\theta}{1-\sin\theta}) + \frac{\sin\theta}{2\cos^2\theta} + C$$

$$= \frac{1}{4}\ln(\frac{1+\frac{x}{\sqrt{x^2+1}}}{1-\frac{x}{\sqrt{x^2+1}}}) + \frac{\frac{x}{\sqrt{x^2+1}}}{\frac{2}{x^2+1}} + C = \frac{1}{4}\ln(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}-x}) + \frac{x\sqrt{x^2+1}}{2} + C$$