1) Let $\mathcal{C}$ be the curve given by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ for $t \in [0, 2]$.

(a) [3 points] Find the equation of the line tangent to $\mathcal{C}$ at the point given by $t = 1$.

Solution. We have $\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$. Then,

$$\mathbf{L}(t) = \mathbf{r}(1) + t \cdot \mathbf{r}'(1) \quad \implies \quad \mathbf{L}(t) = \langle 1, 1, 1 \rangle + t \cdot \langle 1, 2, 3 \rangle = \langle 1 + t, 1 + 2t, 1 + 3t \rangle.$$

□

(b) [3 points] Give a simple [i.e., Calculus 2] integral that gives the arc length of $\mathcal{C}$ for $t \in [0, 2]$. **Do not compute the integral!**

Solution.

$$\int_0^2 \|\mathbf{r}'(t)\| \, \mathrm{d}t = \int_0^2 \sqrt{1 + 4t^2 + 9t^4} \, \mathrm{d}t.$$

□

Continues on the next page!

(c) [5 points] Give a simple [i.e., Calculus 2] integral that gives the work against $\mathbf{F} = \langle x^2, -y^2, z\rangle$ of moving a particle along the curve $\mathcal{C}$ [i.e., the curve given by $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$ for $t \in [0, 2]$; to be clear, we are moving the particle starting at $t = 0$ and ending at $t = 2$]. **Do not compute the integral!**

Solution.

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_0^2 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t \\
&= \int_0^2 \left\langle t^2, -t^4, t^3\right\rangle \cdot \left\langle 1, 2t, 3t^2\right\rangle\,\mathrm{d}t \\
&= \int_0^2 t^2 + t^5\,\mathrm{d}t.
\end{aligned}$$

□

2) Let $f(x, y) = xy$ and $\mathcal{D}$ be the region given by $4x^2 + 9y^2 \leq 32$.

(a) [2 points] Compute the partial derivatives f_x and f_y.

Solution.

$$f_x = y,$$
$$f_y = x.$$

□

(b) [2 points] Find all the critical points of $f(x, y) = xy$ in $\mathbb{R}^2$. [So, not only inside $\mathcal{D}$.]

Solution.

$$f_x = y = 0,$$
$$f_y = x = 0.$$

So, $(x, y) = (0, 0)$ is the only critical point. □

(c) [3 points] For each critical point of the previous part, classify it as local maximum, local minimum, or saddle.

Solution. $f_{xx} = f_{yy} = 0$, $f_{xy} = f_{yx} = 1$. So, the discriminant is $D = 0 \cdot 0 - 1^2 = -1$, hence $(0, 0)$ is a saddle point. □

Continues on the next page!

(d) [6 points] Use Lagrange's Multipliers to find the global maximum and minimum of $f(x,y) = xy$ on the ellipse $4x^2 + 9y^2 = 32$ [the boundary of $\mathcal{D}$].

Solution. We have:

$$\begin{aligned} g(x,y) = 0 \quad &\Longrightarrow \quad 4x^2 + 9y^2 = 32 \\ f_x = \lambda \cdot g_x \quad &\Longrightarrow \quad y = \lambda \cdot 8x \\ f_y = \lambda \cdot g_y \quad &\Longrightarrow \quad x = \lambda \cdot 18y \end{aligned}$$

Note that $x \neq 0$, as otherwise the second equation would give $y = 0$, but $(0,0)$ does not satisfy the first equation. Similarly, $y \neq 0$.

Thus,

$$\frac{y}{8x} = \lambda = \frac{x}{18y} \quad \Longrightarrow \quad 18y^2 = 8x^2.$$

Hence, $y = \pm 2x/3$. Substituting in the first equation we get $8x^2 = 32$, and therefore $x = \pm 2$.

So, we have four candidates: $(2, 4/3)$, $(2, -4/3)$, $(-2, -4/3)$ and $(-2, 4/3)$. Therefore, the global maximum on the ellipse is $8/3$ and the global minimum is $-8/3$. $\square$

Continues on the next page!

(e) [3 points] Find the global maximum and minimum of $f(x, y) = xy$ in $\mathcal{D}$ [the region given by $4x^2 + 9y^2 \leq 32$].

Solution. We know that the global maximum and minimum of f in $\mathcal{D}$ occur either at the boundary or at a critical point in the interior of $\mathcal{D}$. The former, by the previous part, gives that maximum and minimum on the boundary are $8/3$ and $-8/3$. The only critical point in the interior of $\mathcal{D}$ is $(0, 0)$, which gives 0.

So, the global maximum is $8/3$ and the global minimum is $-8/3$. □

3) Let $f(x, y, z) = xy^2 - zx^2$ and $\mathbf{F} = \langle y^2 - 2xz, 2xy, -x^2 \rangle$.

(a) [3 points] Show that $f(x, y, z)$ is the potential of $\mathbf{F}$.

Solution. We have $f_x = y^2 - 2xz$, $f_y = 2xy$, and $f_z = -x^2$. Thus, $\nabla f = \mathbf{F}$, i.e., f is the potential of $\mathbf{F}$. □

(b) [3 points] In what direction from $P = (0, 1, 0)$ does the potential of $\mathbf{F} = \langle y^2 - 2xz, 2xy, -x^2 \rangle$ [as before] increase the most? [Your answer should be a three dimensional vector. **Hint:** Don't let all the terminology confuse you. This is a very simple question.]

Solution. The potential, i.e., $f(x, y, z)$ increases with the highest rate at $\nabla f(P) = \mathbf{F}(P) = \langle 1, 0, 0 \rangle$. □

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(c) [4 points] Let $\mathcal{C}$ be the polygonal path [i.e., made of straight line segments] going from $(0,0,0)$, to $(0,-1,0)$, to $(0,0,2)$ and finally to $(1,1,0)$. Compute $\displaystyle\int_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$.

Solution. We have, since f is the potential of $\mathbf{F}$, that

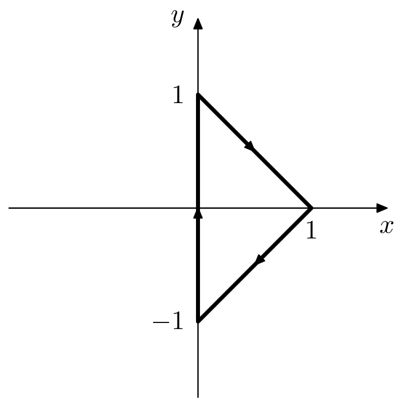
$$\int_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = f(1,1,0) - f(0,0,0) = 1 - 0 = 1.$$

$\square$

4) Let $\mathcal{C}$ be the triangle with vertices $(0, 1)$, $(0, -1)$ and $(1, 0)$, oriented *clockwise*, and $\mathbf{F} = \langle \mathrm{e}^x, \sin(y) - 2x \rangle$.

(a) [3 points] Sketch $\mathcal{C}$. Draw arrows on $\mathcal{C}$ to show the correct orientation.

Solution.



□

(b) [3 points] Compute $\mathrm{curl}_z(\mathbf{F})$. [Remember $\mathbf{F} = \langle \mathrm{e}^x, \sin(y) - 2x \rangle$.]

Solution.

$$\mathrm{curl}_z(\mathbf{F}) = \frac{\partial}{\partial x}(\sin(y) - 2x) - \frac{\partial}{\partial y}(\mathrm{e}^x) = -2 - 0 = -2.$$

□

Continues on the next page!

(c) [6 points] Compute $\oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$. [Remember $\mathcal{C}$ is the triangle with vertices $(0,1)$, $(0,-1)$ and $(1,0)$, oriented *clockwise*, and $\mathbf{F} = \langle \mathrm{e}^x, \sin(y) - 2x \rangle$.]

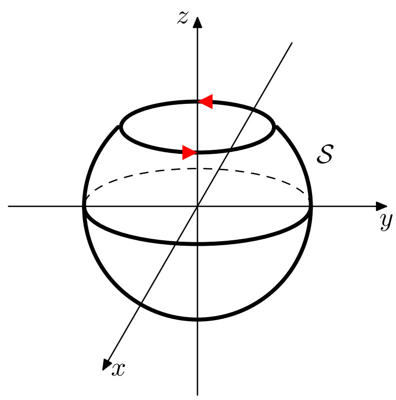
[**Hint:** There is an easy way and a hard way of doing this. If you do it the hard way, you might be pressed on time.]

Solution. Let $\mathcal{D}$ be the *filled* triangle with boundary $\mathcal{C}$. Note that the orientation of $\mathcal{C}$ [clockwise] is the opposite of the boundary orientation! Then, by Green's Theorem:

$$\begin{aligned}
\oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= -\iint_{\mathcal{D}} \mathrm{curl}_z(\mathbf{F})\, \mathrm{d}A \\
&= -\iint_{\mathcal{D}} -2\, \mathrm{d}A \\
&= 2 \cdot \mathrm{area}(\mathcal{D}) \\
&= 2 \cdot 1 = 2.
\end{aligned}$$

□

5) Let $\mathcal{S}$ be the surface given part of the sphere $x^2 + y^2 + z^2 = 13$ with $z \leq 2$ [so we are "chopping" the top of the sphere at height 2 – see figure below], oriented with normal vectors pointing *toward* its center. Let also $\mathbf{F} = \langle y, -x, 0 \rangle$



(a) [2 points] Draw on the picture above an arrow on the curve $\partial\mathcal{S}$ [the *boundary* of the surface $\mathcal{S}$ above] to show the correct boundary orientation.

[**Hint:** The boundary is where $z = 2$.]

(b) [4 points] Give a parametrization $\mathbf{r}(t)$ of $\partial\mathcal{S}$. [Don't forget to give the range of the parameter t.]

Solution. We have, when $z = 2$: $x^2 + y^2 + 4 = 13$, so $x^2 + y^2 = 9$. Hence, $\partial\mathcal{S}$ is a circle of radius 3 on the plane $z = 2$. Thus,

$$\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), 2 \rangle .$$

Since this parametrization goes counterclockwise [when seen from above], it has the correct orientation. $\square$

(c) [3 points] Compute curl($\mathbf{F}$). [Remember: $\mathbf{F} = \langle y, -x, 0 \rangle$.]

Solution.

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = \langle 0, 0, -1 - 1 \rangle = \langle 0, 0, -2 \rangle .$$

$\square$

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(d) [6 points] Compute $\iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S}$. [Remember: $\mathbf{F} = \langle y, -x, 0\rangle$.]

[**Hint:** There is an easy way and a hard way of doing this. If you do it the hard way, you might be pressed on time. Also, if you need to use (b), and could not find it, use the parametrization $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), 1\rangle$ for $t \in [0, \pi]$.]

Solution. Since $\mathcal{S}$ is a bit complicated, it might be easier to use Stokes' Theorem. So:

$$\begin{aligned}
\iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S} &= \oint_{\partial\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\
&= \int_0^{2\pi} \langle 3\sin(t), -3\cos(t), 0\rangle \cdot \langle -3\sin(t), 3\cos(t), 0\rangle \, \mathrm{d}t \\
&= \int_0^{2\pi} -9\sin^2(t) - 9\cos^2(t) \, \mathrm{d}t \\
&= \int_0^{2\pi} -9 \, \mathrm{d}t = -18\pi.
\end{aligned}$$

$\square$

6) Let $\mathcal{W}$ be the box given by $0 \le x \le 1$, $0 \le y \le 2$, and $1 \le z \le 3$ [i.e., $[0,1] \times [0,2] \times [1,3]$] and $\mathbf{F} = \langle xy, yz, y^2 z \rangle$.

(a) [3 points] Compute $\operatorname{div}(\mathbf{F})$.

Solution.

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(y^2 z) = y + z + y^2.$$

□

(b) [6 points] Compute the flux of $\mathbf{F}$ through [or across] $\partial\mathcal{W}$ with the usual boundary orientation.

[**Hint:** There is an easy way and a hard way of doing this. If you do it the hard way, you might be pressed on time.]

Solution. We use Gauss' Theorem:

$$\begin{aligned}
\iint_{\partial\mathcal{W}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\,\mathrm{d}V \\
&= \int_0^1 \left[\int_0^2 \left[\int_1^3 y + y^2 + z \,\mathrm{d}z \right] \mathrm{d}y \right] \mathrm{d}z \\
&= \int_0^1 \left[\int_0^2 2y + 2y^2 + 4 \,\mathrm{d}y \right] \mathrm{d}z \\
&= \int_0^1 4 + \frac{16}{3} + 8 \,\mathrm{d}z \\
&= 12 + \frac{16}{3} = \frac{52}{3}.
\end{aligned}$$

□

7) [5 points] Let $\mathcal{S}$ be the surface given by the parts of the sphere $x^2 + y^2 + z^2 = 9$ with $x \geq 0$ and $z \leq 0$. Give a parametrization of $\mathcal{S}$.

Solution. We use spherical coordinates with $\rho = 3$, $\theta \in [-\pi/2, \pi/2]$ [for $x \geq 0$] and $\phi \in [\pi/2, \pi]$ [for $z \leq 0$]. So, the parametrization is

$$G(u, v) = (3\cos(\theta)\sin(\phi), 3\sin(\theta)\sin(\phi), 3\cos(\phi))$$

with $\theta \in [-\pi/2, \pi/2]$ and $\phi \in [\pi/2, \pi]$. $\square$

8) Let $\mathcal{S}$ be the surface given by the parametrization $G(u,v) = (u, v, u^2 - v^2)$ for $(u,v) \in \mathcal{D}$, where $\mathcal{D}$ is the disc $x^2 + y^2 \leq 4$.

(a) [3 points] Compute the partial derivatives $G_u(u,v)$ and $G_v(u,v)$.

Solution.

$$G_u = \langle 1, 0, 2u \rangle \,,$$
$$G_v = \langle 0, 1, -2v \rangle \,.$$

□

(b) [3 points] Compute $G_u(u,v) \times G_v(u,v)$ [for G_u and G_v found in part (b)].

Solution.

$$G_u \times G_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2u \\ 0 & 1 & -2v \end{vmatrix} = \langle 2u, 2v, 1 \rangle \,.$$

□

(c) [3 points] Give the equation of the plane tangent to $\mathcal{S}$ at the point $(0, 1, -1)$ in the form $ax + by + cz = d$.

Solution. If $G(u,v) = (u, v, u^2 - v^2) = (0, 1, -1)$, then $(u,v) = (0,1)$. The normal vector is then $\langle 2 \cdot 0, 2 \cdot 1, 1 \rangle = \langle 0, 2, 1 \rangle$. So, the tangent plane is:

$$(\langle x, y, z \rangle - \langle 0, 1, -1 \rangle) \cdot \langle 0, 2, 1 \rangle = 0.$$

So,

$$0 \cdot (x - 0) + 2 \cdot (y - 1) + 1 \cdot (z - 1) = 0 \quad \Longrightarrow \quad 2y + z = 3.$$

□

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(d) [6 points] Express the surface area of $\mathcal{S}$ [the same $\mathcal{S}$ as in the previous items] as iterated simple [i.e., Calculus 2] integrals. [In other words, you will set up the integral that gives the area of $\mathcal{S}$, **but not compute it!**]

Solution. The surface area of $\mathcal{S}$ is given by:

$$\begin{aligned}\iint_{\mathcal{S}} 1 \,\mathrm{d}\mathbf{S} &= \iint_{\mathcal{S}} \|\mathbf{N}\| \,\mathrm{d}A \\ &= \iint_{\mathcal{D}} \|G_u \times G_v\| \,\mathrm{d}A \\ &= \iint_{\mathcal{D}} \|\langle 2u, 2v, 1\rangle\| \,\mathrm{d}A \\ &= \iint_{\mathcal{D}} \sqrt{4u^2 + 4v^2 + 1} \,\mathrm{d}A \\ &= \int_0^{2\pi} \left[\int_0^2 \sqrt{4r^2 + 1}\, r \,\mathrm{d}r \right] \mathrm{d}\theta.\end{aligned}$$

□

Continues on the next page!

(e) [6 points] Let now $\mathbf{F} = \langle z, y, xy \rangle$ and $\mathcal{S}$ be as above [given by $G(u,v) = (u, v, u^2 - v^2)$ for $(u,v) \in \mathcal{D}$, where $\mathcal{D}$ is the disc $x^2 + y^2 \leq 4$] oriented with *upward pointing* normal vectors. Express $\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ as a double integral. [In other words, **you do *not* even need to write it as iterated simple integrals**, simply as a *double* (Calculus 3, Chapter 15) area integral, something like $\iint_{\cdots} \cdots \, \mathrm{d}A$. *Don't forget to give the domain of integration and simplify the integrand!*]

Solution. Note that our normal vectors are upward points, since $\langle 2u, 2v, 1 \rangle$ has positive z-coordinate.

$$\begin{aligned}
\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \iint_{\mathcal{D}} \mathbf{F}(G(u,v)) \cdot \mathbf{N}(u,v) \, \mathrm{d}A \\
&= \iint_{\mathcal{D}} \left\langle u^2 - v^2, v, uv \right\rangle \cdot \langle 2u, 2v, 1 \rangle \, \mathrm{d}A \\
&= \iint_{\mathcal{D}} 2u^2 - 2uv^2 + 2v^2 + uv \, \mathrm{d}A.
\end{aligned}$$

Here $\mathcal{D}$ is still the one from the parametrization of $\mathcal{S}$. □