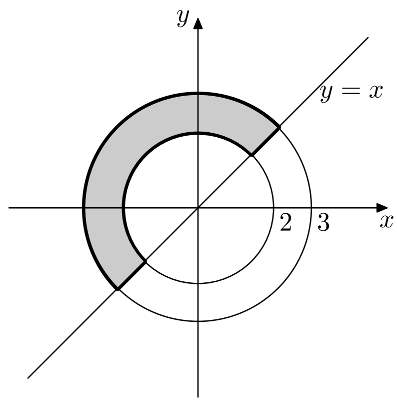
1) Let $\mathcal{D}$ be the region on the plane given by $4 \leq x^2 + y^2 \leq 9$ and $y \geq x$.

(a) [10 points] Sketch $\mathcal{D}$.

Solution.



□

(b) [10 points] Use *polar coordinates* to set up the integral $\iint_{\mathcal{D}} x - y \, \mathrm{d}A$. [You do not need to compute it, just set up as iterated integrals.]

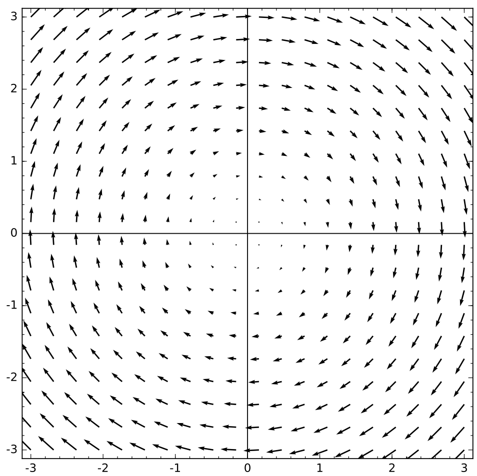
Solution. We have $\theta \in [\pi/4, 5\pi/4]$ and $r \in [2, 3]$:

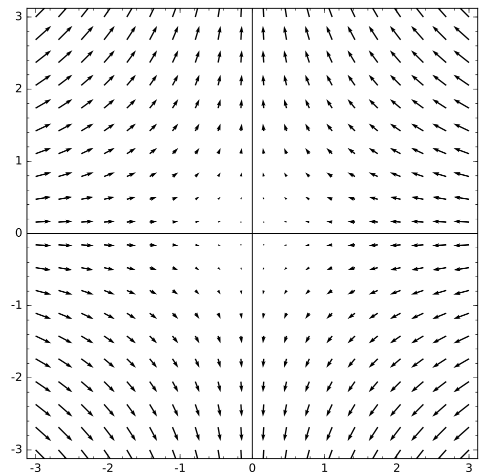
$$\iint_{\mathcal{D}} x - y \, \mathrm{d}A = \int_{\pi/4}^{5\pi/4} \left[\int_{2}^{3} (r\cos(\theta) - r\sin(\theta)) \cdot r \, \mathrm{d}r \right] \mathrm{d}\theta.$$

□

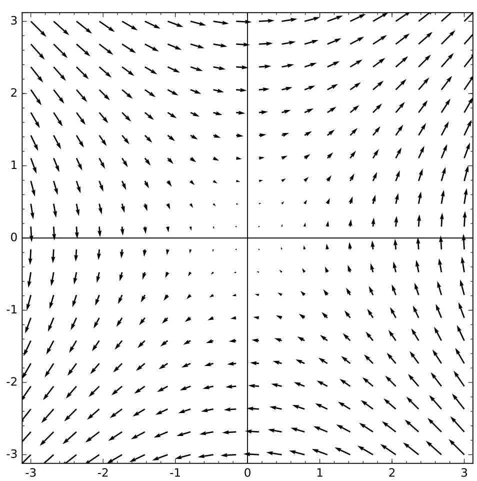
2) [15 points] Match the vector fields to their corresponding plots by writing the corresponding letters by the square containing the plot. (Leave the plots with no match unmarked.)

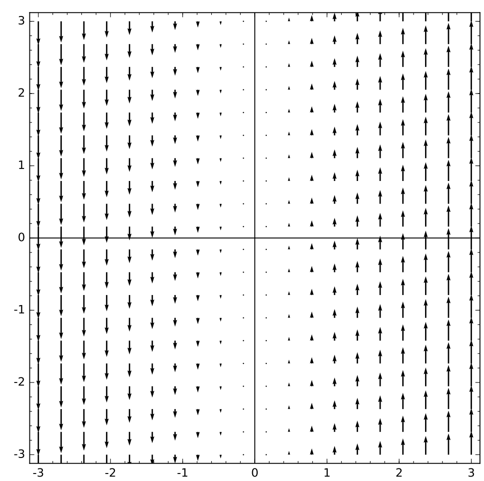
A: $\mathbf{F}(x, y) = \langle 0, y \rangle$ B: $\mathbf{F}(x, y) = \langle -x, y \rangle$

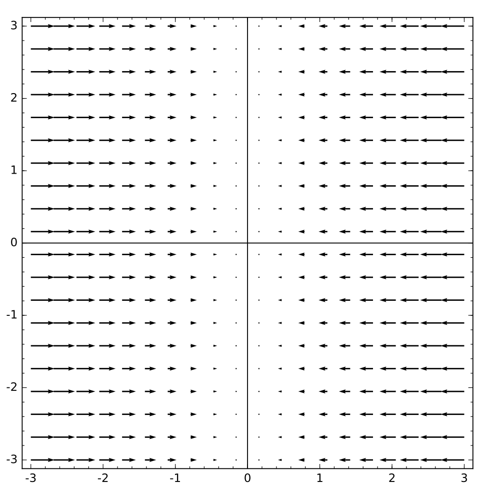


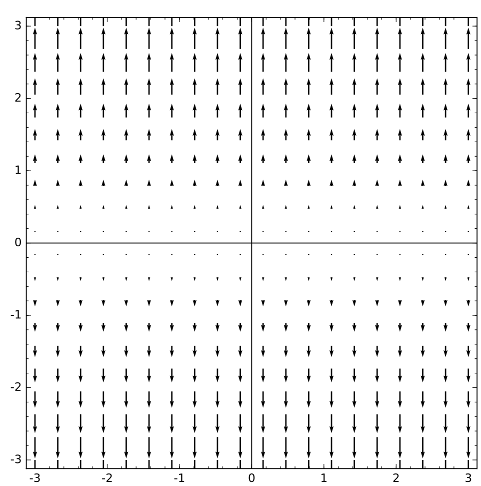


B



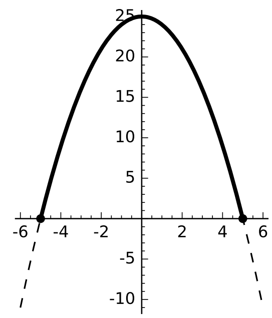






A

3) [15 points] Let $\mathbf{F}(x, y) = \langle x^2 y, y^2 \rangle$ and $\mathcal{C}$ be the piece of the parabola $y = 25 - x^2$ going from $(-5, 0)$ to $(5, 0)$. [See picture below.] Set up [but do not compute!] the vector line integral $\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ as a simple Calculus 2 integral.



Solution. We parametrize $\mathcal{C}$ by $\mathbf{r} = \langle t, 25 - t^2 \rangle$ for $t \in [-5, 5]$. Then,

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_{-5}^{5} \left\langle t^2(25 - t^2), (25 - t^2)^2 \right\rangle \cdot \langle 1, -2t \rangle \; \mathrm{d}t \\ &= \int_{-5}^{5} t^2(25 - t^2) - 2t(25 - t^2)^2 \mathrm{d}t. \end{aligned}$$

□

4) Let $\mathbf{F} = \langle g, h \rangle = \langle 2x + y^2 - \mathrm{e}^{x+3y}, 1 + 2xy - 3\mathrm{e}^{x+3y} \rangle$.

(a) [8 points] Show that $\mathbf{F}$ is a conservative vector field *without computing its potential.*

Solution. First since the domain is $\mathbb{R}^2$, the domain is simply connected (no holes). Then,

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(2x + y^2 - \mathrm{e}^{x+3y}\right) = 2y - 3\mathrm{e}^{x+3y},$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(1 + 2xy + y^2 - 3\mathrm{e}^{x+3y}\right) = 2y - 3\mathrm{e}^{x+3y}.$$

Since these values match and the domain is simply connected, $\mathbf{F}$ is conservative.

□

(b) [9 points] Find the potential of $\mathbf{F}$.

Solution.

$$f(x, y) = \int 2x + y^2 - \mathrm{e}^{x+3y}\,\mathrm{d}x = x^2 + xy^2 - \mathrm{e}^{x+3y} + C_1(y),$$

$$f(x, y) = \int 1 + 2xy - 3\mathrm{e}^{x+3y}\,\mathrm{d}y = y + xy^2 - \mathrm{e}^{x+3y} + C_2(x).$$

So, we can take $C_1(y) = y$ and $C_2(x) = x^2$. Hence $f(x, y) = y + x^2 + xy^2 - \mathrm{e}^{x+3y}$. □

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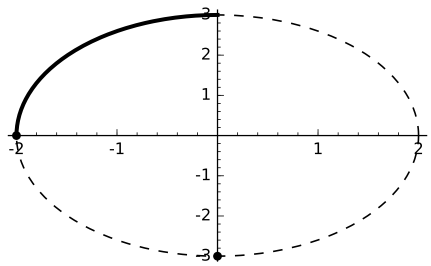
(c) [8 points] Let $\mathcal{C}$ be the part of ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ between $(-2, 0)$ and $(0, 3)$, going *clockwise*. [See picture below.] Compute $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$. [Here it is not enough to simply set it up!]



Solution. Since the field is conservative with potential $f(x, y) = y + x^2 + xy^2 - \mathrm{e}^{x+3y}$, we have

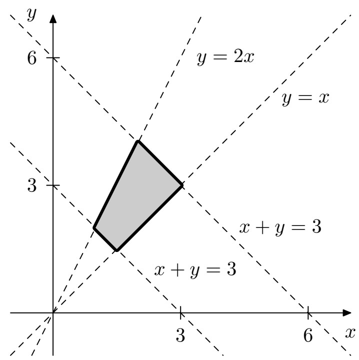
$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = f(0, 3) - f(-2, 0) = 3 - \mathrm{e}^9 - (4 - \mathrm{e}^{-2}) = -1 - \mathrm{e}^9 + \mathrm{e}^{-2}.$$

□

5) Let $\mathcal{D}$ be the region given by $x \leq y \leq 2x$ and $3 \leq x+y \leq 6$ and let $G(u,v) = \left(\frac{u}{v+1}, \frac{uv}{v+1}\right)$.

(a) [6 points] Sketch the region $\mathcal{D}$.

Solution.



□

(b) [6 points] Compute Jac(G), the Jacobian determinant of $G(u,v)$.

Solution.

$$\begin{aligned}
\text{Jac}(G) &= \begin{vmatrix} \frac{1}{v+1} & -\frac{u}{(v+1)^2} \\ \frac{v}{v+1} & \frac{u(v+1)-uv}{(v+1)^2} \end{vmatrix} \\
&= \frac{u}{(v+1)^3} + \frac{uv}{(v+1)^3} \\
&= \frac{u+uv}{(v+1)^3} = \frac{u(v+1)}{(v+1)^3} \\
&= \frac{u}{(v+1)^2}.
\end{aligned}$$

□

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(c) [7 points] Sketch the domain $\mathcal{D}_0$ such that $G(\mathcal{D}_0) = \mathcal{D}$ in the (u, v)-plane.

[**Note:** $\mathcal{D}_0$ is simply the new region of integration when you make the change of variables using $G(u, v)$ above.]

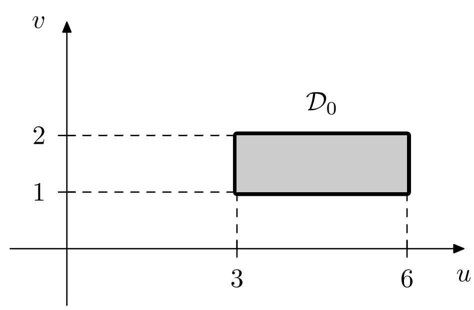
Solution. We have $x(u, v) = u/(v + 1)$ and $y(u, v) = uv/(v + 1)$. Since

$$x(u, v) + y(u, v) = \frac{u}{v+1} + \frac{uv}{v+1} = u$$

and

$$\frac{y(u, v)}{x(u, v)} = \frac{uv}{v+1} \cdot \left(\frac{u}{v+1}\right)^{-1} = v,$$

the conditions $3 \le x + y \le 6$ and $1 \le y/x \le 2$ [as $x > 0$] become $3 \le u \le 6$ and $1 \le v \le 2$.



□

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(d) [6 points] Set up [but do not compute] the integral $\iint_{\mathcal{D}} x/y\, \mathrm{d}A$ *by changing the variables* using $G(u,v)$ above [i.e., $G(u,v) = \left(\frac{u}{v+1}, \frac{uv}{v+1}\right)$].

[**Note:** If you could not do part (c) to find the new domain of integration, use $\mathcal{D}_0 = [1,2] \times [2,3]$ instead.]

Solution.

$$\begin{aligned}
\iint_{\mathcal{D}} \frac{x}{y}\, \mathrm{d}A &= \iint_{\mathcal{D}_0} \frac{x(u,v)}{y(u,v)} \cdot |\operatorname{Jac}(G)|\, \mathrm{d}u\mathrm{d}v \\
&= \iint_{\mathcal{D}_0} \frac{1}{v} \cdot \left| \frac{u}{(v+1)^2} \right| \mathrm{d}u\mathrm{d}v \\
&= \iint_{\mathcal{D}_0} \frac{u}{v(v+1)^2}\, \mathrm{d}u\mathrm{d}v \\
&= \int_3^6 \left[\int_1^2 \frac{u}{v(v+1)^2}\, \mathrm{d}v \right] \mathrm{d}u.
\end{aligned}$$

□