1) [15 points] Let $f(x, y)$ be a function with continuous second order partial derivatives. The table below gives the list of all its critical points, as well as its second order partial derivatives at those points. Fill in the discriminant [also called the Hessian determinant] column and check the corresponding box if the second derivative test gives that the critical point is a local minimum, local maximum, saddle point or if the test is inconclusive.

P (critical point)	$f_{xx}(P)$	$f_{yy}(P)$	$f_{xy}(P)$	discriminant	classification
$(1, 2)$	2	1	-1	1	☑ loc. min. ☐ loc. max. ☐ saddle ☐ inconcl.
$(3, 1)$	-3	-1	2	-1	☐ loc. min. ☐ loc. max. ☑ saddle ☐ inconcl.
$(0, -2)$	1	1	1	0	☐ loc. min. ☐ loc. max. ☐ saddle ☑ inconcl.
$(-3, 0)$	-4	-1	-1	3	☐ loc. min. ☑ loc. max. ☐ saddle ☐ inconcl.
$(-1, 2)$	0	2	-3	-9	☐ loc. min. ☐ loc. max. ☑ saddle ☐ inconcl.

2) The temperature at a point (x, y) on the plane is give by the function $T(x, y) = 2x^2 + y^2$.

(a) [10 points] In what direction from the point $(1, 1)$ does the temperature *decrease* the most? [Your answer should be a vector giving the direction on the xy-plane.]

Solution. The direction of greatest decrease is given by $-\nabla T(P)$. We have $\nabla T(x, y) = \langle 4x, 2y\rangle$, and so, the direction is $-\nabla T(1, 1) = -\langle 4, 2\rangle = \langle -4, -2\rangle$. $\square$

(b) [10 points] What is the rate of change of the temperature if you move from $(1, 1)$ again, but in the direction of the vector $\langle -1, 1\rangle$?

Solution. From part (a), we have that $\nabla T(x, y) = \langle 4x, 2y\rangle$. The unitary vector in the given direction is $\vec{u} = \langle -1, 1\rangle / \| \langle -1, 1\rangle \| = \left\langle -1/\sqrt{2}, 1/\sqrt{2}\right\rangle$. So, the rate of change is

$$\begin{aligned} D_{\vec{u}}T(1,1) &= \nabla T(1,1) \cdot \vec{u} \\ &= \langle 4, 2\rangle \cdot \left\langle -1/\sqrt{2}, 1/\sqrt{2}\right\rangle \\ &= -2/\sqrt{2} = -\sqrt{2}. \end{aligned}$$

$\square$

3) Let $f(x, y) = 5 + 4xy - x^2 - y^2$.

(a) [8 points] Find all the critical points of $f(x, y)$. [*No need to classify them (as local maximum, minimum or saddle)!*]

Solution. We have: $f_x(x, y) = 4y - 2x$ and $f_y(x, y) = 4x - 2y$. So, we have the system:

$$\begin{aligned} -2x + 4y &= 0 \\ 4x - 2y &= 0. \end{aligned}$$

The first one gives us $x = 2y$ and substituting it in the second we get $6y = 0$, and hence $y = 0$. So, $x = 2 \cdot 0 = 0$. Hence, the only critical point is $(0, 0)$.

$\square$

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(b) [12 points] Use Lagrange multipliers to find the possible extrema of $f(x, y) = 5 + 4xy - x^2 - y^2$ [same $f(x, y)$] with the constraint $x^2 + y^2 = 4$. [*No need to classify them (as local maximum, minimum or saddle)!*]

Solution. We have the system:

$$
\begin{aligned}
x^2 + y^2 &= 4 \\
-2x + 4y &= \lambda 2x \\
4x - 2y &= \lambda 2y.
\end{aligned}
$$

Note that if $x = 0$, then the second equation gives us that $y = 0$, but $(0, 0)$ does not satisfy the first equation. So, $x \neq 0$.

Similarly, if $y = 0$, then the third equation gives us that $x = 0$, but, again, $(0, 0)$ does not satisfy the first equation. So, $y \neq 0$ also.

Then, solving for λ in the last two equations we get:

$$2\,\frac{y}{x} - 1 = \lambda = 2\,\frac{x}{y} - 1.$$

So, $x/y = y/x$, i.e., $x^2 = y^2$, which means $x = \pm y$. Substituting in the first equation we get $2x^2 = 4$, so $x = \pm\sqrt{2}$.

Thus, we have 4 possible extrema: $(\sqrt{2}, \sqrt{2})$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$, $(-\sqrt{2}, -\sqrt{2})$. $\square$

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(c) [5 points]Find the *global* maximum and minimum of $f(x,y) = 5 + 4xy - x^2 - y^2$ [the same $f(x,y)$] on $x^2 + y^2 \leq 4$ and at which points they occur.

Proof. We just need to collect the data from the previous parts:

P	$f(P)$	classification
$(0,0)$	5	neither
$(\sqrt{2}, \sqrt{2})$	17	maximum
$(-\sqrt{2}, -\sqrt{2})$	17	maximum
$(\sqrt{2}, -\sqrt{2})$	-15	minimum
$(-\sqrt{2}, \sqrt{2})$	-15	minimum

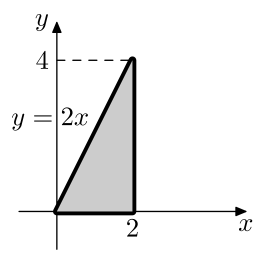
$\square$

4) Consider the integral

$$\int_0^4 \left[\int_{y/2}^2 (x^2-1)^{2018}\,\mathrm{d}x\right]\,\mathrm{d}y.$$

(a) [10 points] Draw the region of integration for the double integral it represents.

Solution.



□

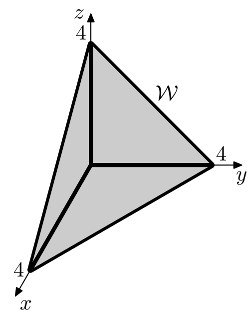
(b) [10 points] Express the integral in the opposite order. **Do *not* compute the integral!**

Solution.

$$\int_0^2 \left[\int_0^{2x} (x^2-1)^{2018}\,\mathrm{d}y\right]\,\mathrm{d}x.$$

□

5) [20 points] Let $\mathcal{W}$ be the solid tetrahedron [i.e., a pyramid with four faces] bounded by the planes $x = 0$ [the yz-plane], $y = 0$ [the xz-plane], $z = 0$ [the xy-plane] and $x + y + z = 4$. [See the image below]. Compute $\displaystyle\iiint_{\mathcal{W}} \mathrm{e}^z \, \mathrm{d}V$.



Solution. Let $\mathcal{D}$ be the triangle on the xy-plane. Then:

$$\begin{aligned}
\iiint_{\mathcal{W}} \mathrm{e}^z \, \mathrm{d}x\mathrm{d}y\mathrm{d}z &= \iint_{\mathcal{D}} \left[\int_0^{4-x-y} \mathrm{e}^z \, \mathrm{d}z\right] \mathrm{d}x\mathrm{d}y \\
&= \iint_{\mathcal{D}} [\mathrm{e}^z]_{z=0}^{z=4-x-y} \, \mathrm{d}x\mathrm{d}y \\
&= \iint_{\mathcal{D}} \mathrm{e}^{4-x-y} - 1 \, \mathrm{d}x\mathrm{d}y \\
&= \int_0^4 \left[\int_0^{4-x} \mathrm{e}^{4-x-y} - 1 \, \mathrm{d}y\right] \mathrm{d}x \\
&= \int_0^4 \left[-\mathrm{e}^{4-x-y} - y\right]_{y=0}^{y=4-x} \mathrm{d}x \\
&= \int_0^4 [-1 - (4 - x)] - \left[-\mathrm{e}^{4-x}\right] \mathrm{d}x \\
&= \int_0^4 -5 + x + \mathrm{e}^{4-x} \mathrm{d}x \\
&= \left[-5x + x^2/2 - \mathrm{e}^{4-x}\right]_{x=0}^{x=4} \\
&= [-20 + 8 - 1] - \left[-\mathrm{e}^4\right] \\
&= -13 + \mathrm{e}^4
\end{aligned}$$

□