Final

This is a take-home exam: You cannot talk to *anyone* (except me) about *anything* about this exam and you can only look at *our* book (Walker), class notes and solutions to *our* HW problems *posted by me* or done by yourself. No other reference, including the Internet. Failing to follow these instructions will result in a zero for the exam. Moreover, I will report the incident to the university and do all in my power to get the maximal penalty for the infraction.

Due date: noon on Wednesday (04/30). If you cannot bring it to my office (slide it under the door if I'm not in), a scanned/typed copy by e-mail would be OK.

1) [20 points] Let F be a field of characteristic p [and $p \neq 0$, a prime integer] and assume that F is *perfect* [i.e., for each $a \in F$, there is a unique $b \in F$ such that $a = b^p$, i.e., each element of F has a unique p-th root in F]. Let $\bar{F}$ be a fixed algebraic closure of F.

(a) Let $\alpha \in \bar{F}$ *separable* over F and $\beta \in \bar{F}$ such that $\beta^p = \alpha$ [i.e., β is *the* root of $x^p - \alpha$ in $\bar{F}$]. Show that $F[\beta] = F[\alpha]$. [**Hint:** Clearly $F[\beta] \supseteq F[\alpha]$. Let $f \stackrel{\text{def}}{=} \min_F(\alpha)$ and $g \stackrel{\text{def}}{=} \min_F(\beta)$. Use f to find g, showing that $\deg f = \deg g$. Of course, you will need to use the fact that F is perfect.]

(b) Prove that $F[\alpha]$ above is also perfect.

(c) Let K be a finite separable extension of F. Show that K is perfect.

[**Note:** It actually follows from the last item, and a little effort, that if K/F is simply finite, then K is perfect. But you don't have to do it here.]

2) [20 points] Let $f = (x^4 - 1)(x^2 - 3) \in \mathbb{Q}[x]$. What is the Galois group isomorphic to? [Of course, *justify*! Note I am not asking for diagrams, intermediate fields, action on generators, etc. Only a description of the Galois group as a familiar group, such as $(\mathbb{Z}/5\mathbb{Z}) \times D_7 \times S_4$, or something like that.]

3) [20 points] Let R be a commutative ring with 1, A, B and C be R-modules and $\phi : A \to B$ and $\psi : B \to C$ homomorphisms such that $\psi \circ \phi : A \to C$ is an *isomorphism*. Prove that $B = \operatorname{im} \phi \oplus \ker \psi$. [**Hint:** Prove that if $b \in B$, then $\psi(b) = \psi(\phi(a))$ for some $a \in A$. What can we say about $b - \phi(a)$? Note that $\phi(a) \in \operatorname{im} \phi$, of course.]

4) [20 points] Let R be a PID, M an R-module such that $M = T_1 \oplus F_1 = T_2 \oplus F_2$, where T_i is torsion and F_i is free. [Both are *internal* direct sums.]

(a) Prove that $T_1 = T_2$ [this is an *equality!*] and $F_1 \cong F_2$ [this is an isomorphism].

(b) Give an example where $F_1 \cong F_2$, but $F_1 \neq F_2$. [**Hint:** You can let $R = \mathbb{Z}$, $M = (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$, $T_1 = T_2 = (\mathbb{Z}/2\mathbb{Z}) \oplus \{0\}$, and $F_1 = \{0\} \oplus \mathbb{Z}$, and then find a F_2 that does the job, i.e., $F_2 \neq F_1$, $F_2 \cong \mathbb{Z}$ [hence cyclic], but $M = T_2 \oplus F_2$.]

5) [20 points] Let V be a finite dimensional vector space over F and $\phi : V \to V$ a linear transformation [i.e., homomorphism]. Then, we can give an $F[x]$-module structure to V by,

$$(a_n x^n + \cdots a_1 x + a_0) \cdot v \stackrel{\text{def}}{=} a_n \phi^n(v) + \cdots + a_1 \phi(v) + a_0 v$$

[where $\phi^n = \phi \circ \phi \circ \cdots \circ \phi$], in other words, $x \cdot v \stackrel{\text{def}}{=} \phi(v)$. [You *don't* need to show that this indeed gives an $F[x]$-module structure.]

Assume that, as $F[x]$-modules, we have $V \cong F[x]/(f)$, for some $f \in F[x]$. Show that if $a \in F$ is such that $f(a) = 0$, then there exists $v \in V \setminus \{0\}$ such that $\phi(v) = a \cdot v$ [i.e., a is an *eigenvalue* of ϕ and v is an *eigenvector* of ϕ associated to a]. [**Hint:** $\phi(v) = a \cdot v$ iff $(x - a) \cdot v = 0$.]

[**Note:** The converse is also true.]

6) [20 points] Let V be a finite dimensional vector space, $\{v_1^*, \ldots, v_n^*\}$ and basis of V^* and $\{v_1^{**}, \ldots, v^{**}\}$ its dual basis [so, a basis of V^{**}]. Let also

$$\eta : V \to V^{**}$$

the *natural isomorphism* [i.e., $\eta(v)(v^*) \stackrel{\text{def}}{=} v^*(v)$, i.e., $\eta(v)$ is the evaluation at v map].

(a) Prove that the original basis $\{v_1^*, \ldots, v_n^*\}$ is the dual basis of $\{v_1, \ldots, v_n\}$, where $v_i \stackrel{\text{def}}{=} \eta^{-1}(v_i^{**})$.

(b) Let $v_1^*, v_2^* \in V^* \setminus \{0\}$. Prove that if $\ker v_1^* = \ker v_2^*$, then $v_2^* = a v_1^*$ for some $a \in F$ [i.e., $\{v_1^*, v_2^*\}$ is linearly dependent]. [**Hint:** Assume linearly independent and use part (a).]