FINAL

This is a take-home exam: You cannot talk to *anyone* (except me) about *anything* about this exam and you can only look at *our* book (Bona) and class notes. *No other reference is allowed*, including the Internet. Failing to follow these instructions will result in a zero for the exam. Moreover, I will report the incident to the university and do all in my power to get the maximal penalty for the infraction.

As you will have a lot of time, I expect the solutions to be well written and clearly explained.

If you need clarifications on any statement, please use the "Q&A (Math Related)" forum on Blackboard.

Due date: 5pm on Monday (05/05) *by e-mail*. Please send as a PDF and make sure your scanned/typed exam is clear and legible. *If I can't read your solution, I will give you zero for the problem!*

Always show work!

1) [12 points] A student has 5 different combinatorics books, 8 different algebra books, 4 different analysis books and 6 different geometry books. In how many ways can these books be arranged on a shelf so that all the books of the same topic are together?

2) [12 points] How many rearrangements of the alphabet [26 letters] are such that the vowels, A, E, I, O and U, do *not* appear in order. [Note that there *can* be letters between the vowels. But we don't want for A to come before E, then E before I, etc.]

3) [12 points] A student has to work 12 hours a week [7 days]. If he can only work for an integer number of hours a day [so "one and a half hour on Monday" is *not* allowed], he does not need to work every single day [so, zero hours in a day *is* allowed], but he has to work on Mondays, Wednesdays and Fridays [for at least one hour]. In how many ways can he plan a work week? [Note that the times when he works is irrelevant, we only car about how many hours he works on each day of the week.]

4) [12 points] Show that

$$\sum_{i=0}^{20}\sum_{j=0}^{20-i} \frac{n!}{i!\cdot j!\cdot (n-i-j)!}\cdot 2^i \cdot 3^j = 6^{20}.$$

[**Hint:** $i + j + (n - i - j) = n$.]

5) [13 points] In how many permutations on n-elements [$n \geq 4$] do we have 1 and 2 in the same cycle, but 3 and 4 *not* in the same cycle?

6) [13 points] How many arrangements of $(1, 1, 2, 2, 3, 3, 4, 5, 6, 7)$ have *no* pair of equal digits in consecutive spots.

7) [13 points] Let a_n, for $n \geq 2$, be the numbers of sequences of 0's and 1's with at least one instance of two consecutive 0's. Find a recurrence relation for the a_n's. [That means, a formula for a_n which depends on the previous a_i's. For instance, something like $a_n = a_{n-1}+2n$ or $a_n = 3a_{n-1} - 2a_{n-3}$.]

8) [13 points] Let a_n be such that $a_0 = 1$ and $a_n = 2a_{n-1} + n$. Find a closed formula [no summation] for a_n.