May 2nd, 2013

Math 307

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Spring 2013

Name: ....

Student ID (last 6 digits): XXX- ....

Final

You have two hours to complete the exam. Do all work on this exam, i.e., on the page of the respective assignment. Indicate clearly when you continue your solution on the back of the page or another part of the exam.

Write your name and the last six digits of your student ID number on the top of this page. Check that no pages of your exam are missing. This exam has 10 questions and 12 printed pages (including this one and two pages for scratch work in the end).

No books, notes or calculators are allowed on this exam!

Show all work! Even correct answers without work may result in point deductions. Also, **points will be taken from messy solutions**. Remember you also be graded on how well written your proofs are.

Good luck!

Question	Max. Points	Score
1	8	
2	8	
3	10	
4	10	
5	10	
6	12	
7	10	
8	12	
9	10	
10	10	
Total	100	

1) [8 points] Rewrite the statement [about real numbers]:

$$\neg[\forall x \in \mathbb{R},\ \exists y \in \mathbb{N} \text{ st } [(x \geq y) \to ((x + y > 0) \wedge (x = y + 2))]]$$

as a positive statement [without the "$\neg$"symbol].

2) [8 points] Fill the truth table below.

P	Q	R	$P \wedge Q$	$(\neg Q) \vee R$	$(P \wedge Q) \to ((\neg Q) \vee R)$
T	T	T			
F	T	T			
T	T	F			
F	T	F			

3) [10 points] Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

4) [10 points] Let $\mathcal{F}$ and $\mathcal{G}$ be a families of sets. Prove that $\bigcap(\mathcal{F} \cup \mathcal{G}) = (\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G})$.

5) [10 points] Let A be a set with partial order R and $a \in A$ the smallest element of A. Show that A has a unique minimal element. [What could this element be? In fact, we did this in class.]

6) [12 points] Given $n \in \{1, 2, 3, 4, \ldots\}$, let $(0, 1/n)$ be [as usual in Calculus] the open interval of $\mathbb{R}$ given by $(0, 1/n) = \{x \in \mathbb{R} \,:\, 0 < x < 1/n\}$.
Let

$$\begin{aligned}\mathcal{F} &= \{\{0\}\} \cup \{(0, 1/n) \,:\, n \in \{1, 2, 3, 4, \ldots\}\} \\ &= \{\{0\}, (0, 1), (0, 1/2), (0, 1/3), (0, 1/4), \ldots\},\end{aligned}$$

and consider the partial order in $\mathcal{F}$ given by containment [as usual for sets].

(a) Show that $\{0\}$ is a minimal element of $\mathcal{F}$.

(b) Show that for any $n \in \{1, 2, 3, \ldots\}$, $(0, 1/n)$ is *not* a minimal element of $\mathcal{F}$.

(c) Show that $\mathcal{F}$ has no smallest element. [**Hint:** Remember that if $A \in \mathcal{F}$ is a smallest element, then it is also a minimal element.]

[**Note:** This shows that a set can have only one minimal element, but no smallest element.]

7) [10 points] Let R be the equivalence relation on $\mathbb{R}$ given by aRb if $(a-b) \in \mathbb{Z}$. [You do *not* need to prove it is an equivalence relation.]

(a) Show that $[0]_R = \mathbb{Z}$. [Remember that $[0]_R$ is the equivalence class of 0 with respect to the relation R given above.]

(b) Find a real number x with $0 \leq x < 1$, such that $[2.31]_R = [x]_R$.

8) [12 points] Let R be an equivalence relation on a set A.

(a) Show that both $\mathrm{Ran}(R)$ [the range of R] and $\mathrm{Dom}(R)$ [the domain of R] are equal to A.

(b) Show that R^{-1} [the inverse relation] is equal to R.

(c) Show that $R \circ R$ [the composition] is also equal to R.

9) [10 points] Prove that for $n \geq 0$ we have

$$0 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}.$$

10) [10 points] Remember that the Fibonacci sequence is given by:

$$F_0 = 0, \qquad F_1 = 1,$$

$$F_n = F_{n-2} + F_{n-1}, \quad \text{for } n \geq 2.$$

Consider now the recursively defined sequence given by

$$a_0 = 0, \qquad a_1 = 1, \qquad a_2 = 1,$$

$$a_n = \frac{1}{2}a_{n-3} + \frac{3}{2}a_{n-2} + \frac{1}{2}a_{n-1}, \quad \text{for } n \geq 3.$$

Prove that $a_n = F_n$ for all $n \geq 0$.

[**Hint:** $\frac{3}{2}a_{n-2} = \frac{1}{2}a_{n-2} + a_{n-2}$.]

Scratch:

Scratch: