1) [15 points] Suppose that A, B and C are sets such that $A \subseteq B \setminus C$. Show that $A \cap C = \varnothing$.

Proof. Suppose that $x \in A \cap C$. [We need to derive a contradiction!] Then $x \in A$ and $x \in C$. But, since $x \in A$ and $A \subseteq B \setminus C$, we have that $x \in B \setminus C$, i.e., $x \in B$ and $x \notin C$. But we also have $x \in C$, and hence we have a contradiction. □

2) [15 points] Let B be a set and $\mathcal{F} \subseteq \mathscr{P}(B)$. Show that $\bigcup \mathcal{F} \subseteq B$.

Proof. Let $x \in \bigcup \mathcal{F}$. So, there is $A \in \mathcal{F}$ such that $x \in A$. Since $\mathcal{F} \subseteq \mathscr{P}(B)$, we have that $A \in \mathscr{P}(B)$, i.e., $A \subseteq B$. But, $x \in A$ and $A \subseteq B$ implies that $x \in B$.
Since $x \in \bigcup \mathcal{F}$ was arbitrary, we have that $\bigcup \mathcal{F} \subseteq B$. □

3) [15 points] Let x be an integer. Show that there exists an integer k such that either $x^2 = 4k$ or $x^2 = 4k + 1$. [**Hint:** Remember that x is even if there is an integer n such that $x = 2n$ and it is odd if there is an integer n such that $x = 2n + 1$.]

Proof. We break the proof in cases: x even and x odd.
Suppose first that x is even. Then, $x = 2n$ for some integer n. Therefore, $x^2 = (2n)^2 = 4n^2$. Since n^2 is an integer [as it is a square of an integer] we have that $x^2 = 4k$ for some integer k, namely $k = n^2$.
Suppose now that x is odd. Then, $x = 2n + 1$ for some integer n. Then, $x^2 = (2n + 1)^2 = 4n^2 + 4n + 1 = 4(n^2 + n) + 1$. Since n is an integer, so is $n^2 + n$. Thus, $x^2 = 4k + 1$, where $k = n^2 + n$ is an integer.

□

4) [15 points] Let A, B, C and D be *non-empty* sets. Prove that if $A \times B \subseteq C \times D$ then $A \subseteq C$ and $B \subseteq D$.

Proof. Since A and B are not empty, there is $a \in A$ and $b \in B$. [We need to show that $a \in C$ and $b \in D$, as those were arbitrary.] Then, $(a, b) \in A \times B$, by definition of Cartesian product. Since $A \times B \subseteq C \times D$, this means that $(a, b) \in C \times D$. Then, by definition of Cartesian product again, we have that $a \in C$ and $b \in D$.

Note: The fact that the sets were not empty is essential here! For example, if $A = \{1, 2\}$, $B = \varnothing$, $C = \{1\}$ and $D = \varnothing$, then $A \times B = \varnothing \subseteq \varnothing = C \times D$, but $\{1, 2\} = A \nsubseteq C = \{1\}$.

□

5) Let R be the relation on $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$:

$$R = \{(a, b) \in \mathbb{N} \times \mathbb{N} \ : \ b = 3a\}.$$

(a) [3 points] What is the domain of R? No need to justify.

Solution. The domain is [all of] $\mathbb{N}$, as for every $n \in \mathbb{N}$, we have that $3n \in \mathbb{N}$ and so $(n, 3n) \in R$. □

(b) [3 points] What is the range of R? No need to justify.

Solution. The range is $S = \{0, 3, 6, 9, 12, \ldots\}$, i.e., non-negative multiples of 3. Indeed, if $m \in \text{Ran}(R)$, then there is $n \in \mathbb{N}$ such that $(n, m) \in R$. Hence, $m = 3n$ and so $m \in S$. Therefore, $\text{Ran}(R) \subseteq S$.

Conversely, if $m \in S$, then $m = 3n$ for some $n \in \mathbb{N}$. Thus, $(n, m) \in R$ and hence $m \in \text{Ran}(R)$.

Thus, $S = \text{Ran}(R)$. □

(c) [3 points] What is the range of $R \circ R$? No need to justify.

Solution. It is $\{0, 9, 18, 27, \ldots\}$, i.e., multiples of 9. [Can you prove it?] □

(d) [3 points] Give three different elements of R^{-1}. No need to justify.

Solution. Any three elements of the form $(3n, n)$, where $n \in \mathbb{N}$. For example, $(0, 0)$, $(3, 1)$ and $(6, 2)$. □

(e) [13 points] Check if R is each reflexive, symmetric and transitive? Justify each answer with proofs [in the affirmative case] or counter-examples [in the negative case].

Solution. It is not reflexive, as, for instance, $(1, 1) \notin R$ [as $1 \neq 3 = 3 \cdot 1$].

It is not symmetric, as, for instance, $(1, 3) \in R$, but $(3, 1) \notin R$ [as $1 \neq 9 = 3 \cdot 3$].

It is not transitive, as, for instance, $(1, 3)$ and $(3, 9)$ are in R, but $(1, 9)$ is not [as $9 \neq 3 = 3 \cdot 1$]. □

6) [15 points] Suppose that R_1 and R_2 are symmetric relations on A. Prove that $R_1 \cup R_2$ is also symmetric.

Proof. Let $(x, y) \in R_1 \cup R_2$. [We need to prove that $(y, x) \in R_1 \cup R_2$.] This means that either $(x, y) \in R_1$ or $(x, y) \in R_2$. Since both are symmetric, we have that either $(y, x) \in R_1$ [in the case that $(x, y) \in R_1$] or $(y, x) \in R_2$ [in the case that $(x, y) \in R_2$]. Thus, [by definition of union] we have that $(y, x) \in R_1 \cup R_2$. □