1) [5 points] Let $f(x) = \dfrac{(x-1)(x+2)(x^2-4x+3)}{x}$. Find where $f(x) = 0$, $f(x) > 0$, and $f(x) < 0$.

Solution. We have that $x^2 - 4x + 3 = (x-1)(x-3)$, so $f(x) = \dfrac{(x-1)^2(x+2)(x-3)}{x}$. Then, $f(x) = 0$ at $x = -2, 1, 3$. We have $f(x) > 0$ for x in $(-2, 0)$ or $(3, \infty)$, and $f(x) < 0$ for x in $(-\infty, -2)$, $(0, 1)$, or $(1, 3)$.

$\square$

2) Compute the following limits.

(a) [5 points] $\displaystyle\lim_{x \to 1^+} \frac{x^3 + 2x - 4}{x^2 - x}$

Solution. Since we get "$-1/0$" we must have some kind of infinity. Since the numerator is negative and the denominator is positive for $x > 1$, we get $\displaystyle\lim_{x \to 1^+} \frac{x^3 + 2x - 4}{x^2 - x} = -\infty$.

$\square$

(b) [7 points] $\displaystyle\lim_{x \to 0} \frac{\mathrm{e}^x - x - 1}{x^2}$.

Solution.

$$
\begin{aligned}
\lim_{x \to 0} \frac{\mathrm{e}^x - x - 1}{x^2} &= \lim_{x \to 0} \frac{\mathrm{e}^x - 1}{2x} && \text{[L'Hospital's Rule]} \\
&= \lim_{x \to 0} \frac{\mathrm{e}^x}{2} && \text{[L'Hospital's Rule]} \\
&= \frac{1}{2}.
\end{aligned}
$$

$\square$

(c) [8 points] $\lim_{x\to\infty} x^2 \sin\left(\frac{1}{4x^2}\right)$.

Solution.

$$\begin{aligned}
\lim_{x\to\infty} x^2 \sin\left(\frac{1}{4x^2}\right) &= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{4}x^{-2}\right)}{x^{-2}} \\
&= \lim_{x\to\infty} \frac{\cos\left(\frac{1}{4}x^{-2}\right)\cdot\left(-\frac{1}{2}x^{-3}\right)}{-2x^{-3}} && \text{[L'Hospital's Rule]} \\
&= \lim_{x\to\infty} \frac{\cos\left(\frac{1}{4}x^{-2}\right)\cdot\left(-\frac{1}{2}\right)}{-2} \\
&= \frac{1}{4}\lim_{x\to\infty} \cos\left(\frac{1}{4}x^{-2}\right) = \frac{1}{4}
\end{aligned}$$

□

3) [10 points] If $f(x) = \cos(x)^{x/(e^x+1)}$, compute the derivative $f'(x)$.

Solution. We use logarithmic derivative: $\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$, and hence, $f'(x) = f(x) \cdot \frac{d}{dx}(\ln(f(x)))$. Now:

$$\begin{aligned}
\frac{d}{dx}(\ln(f(x)) &= \frac{d}{dx}\left(\frac{x}{e^x+1}\cdot\ln(\cos(x))\right) \\
&= \frac{d}{dx}\left(\frac{x}{e^x+1}\right)\cdot\ln(\cos(x)) + \frac{x}{e^x+1}\cdot\frac{d}{dx}(\ln(\cos(x))) \\
&= \frac{1\cdot(e^x+1) - x\cdot e^x}{(e^x+1)^2}\cdot\ln(\cos(x)) + \frac{x}{e^x+1}\cdot\frac{1}{\cos(x)}\cdot(-\sin(x))).
\end{aligned}$$

So,

$$f'(x) = \cos(x)^{x/(e^x+1)}\cdot\left(\frac{1\cdot(e^x+1) - x\cdot e^x}{(e^x+1)^2}\cdot\ln(\cos(x)) + \frac{x}{e^x+1}\cdot\frac{1}{\cos(x)}\cdot(-\sin(x)))\right).$$

□

4) [10 points] Consider the curve given by the equation $x^3 + y^4 - y - 1 = 0$ and the curve given by the parametric equations $x = (t+1)\mathrm{e}^t$, $y = \arcsin(t^2) + 1$. Are the tangent lines at the point $(1, 1)$ [which is indeed a point in both curves] orthogonal?

[**Hint:** The second curves passes through $(1, 1)$ when $t = 0$. Also, if you cannot find the tangent lines, you can describe how you'd find if they are perpendicular for a *little* partial credit.]

Solution. For the first equation, we have, taking derivatives:

$$3x^2 + 4y^3 \cdot y' - y' = 0.$$

So, $y' = -3x^2/(4y^3 - 1)$. So, the tangent line at $(1, 1)$ has slope -1.

For the parametrized curve we get

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 \cdot \mathrm{e}^t + (t+1) \cdot \mathrm{e}^t = (t+2)\mathrm{e}^t, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{\sqrt{1-t^4}} \cdot 2t.$$

Hence,

$$y' = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}$$

and at $t = 0$ we get $y' = 0/2 = 0$.

Since $-1 \cdot 0 = 0 \neq -1$, the tangent lines are not orthogonal. □

5) [10 points] A particle moves along a straight line with position [measured as the distance to a fixed point] at time t given by $s(t) = t^4/2 + t^3 - 6t^2$. [Units can be taken to be meters for distance and seconds for time.] For $t \in [0, 4]$ only, when was the velocity of the particle maximal and when was it minimal?

[**Note:** A negative velocity means that the particle is moving backwards. We do consider a negative velocity to be smaller than any positive velocity.]

Solution. We have $v(t) = s'(t) = 2t^3 + 3t^2 - 12t$. To find the when the global maximum and global minimum of $v(t)$ occur, we must find the critical numbers. We have $v'(t) = 6t^2 + 6t - 12 = 6(t^2 + t - 2) = 6(t-1)(t+2)$. [Note that $v'(t)$ is the acceleration.] So, $v'(t) = 0$ for $t = -2, 1$. [We do not consider $t = -2$ and $-2 \notin [0, 4]$.] Then, we have $v(0) = 0$, $v(1) = -7$, $v(4) = 128$. So, the maximum velocity occurred at $t = 4$ and the minimal occurred at $t = 1$. □

6) [10 points] You want to build a box of volume $10\,\mathrm{ft}^3$ and with a *square bottom.* The cost for the material to build the bottom, sides, and top cost \$4, \$2, and \$1 per square foot respectively. Find the dimensions of the box of minimal cost, *as well as* the cost to build such box.

Solution. Let l be the length of the side of the square base, and h be the height of the box. Then, $l^2 \cdot h = 10$, and hence $h = 10/l^2$.

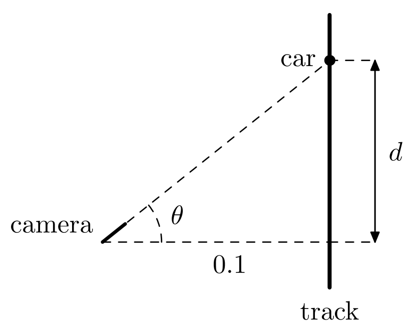
The cost to build the box is $C = 4l^2 + 4 \cdot 2 \cdot lh + 1 \cdot l^2 = 5l^2 + 8lh$. Since $h = 10/l^2$, we get $C = 5l^2 + 80/l$. [Note that $l \in (0, \infty)$.]

We have $C' = 10l - 80/l^2 = (10l^3 - 80)/l^2$. So, $C' = 0$, implies $10l^3 - 80 = 0$, or $l^3 = 8$, i.e., $l = 2$. Analyzing the sign of C', we see that C' is negative in $(0, 2)$ and positive in $(2, \infty)$. Thus, $l = 2$ is the global minimum [for $l \in (0, \infty)$].

So, the dimensions of the box are $l = 2$ and $w = 10/4 = 2.5$ [i.e., the box is $2 \times 2 \times 2.5$] and the cost is then $C = 5 \cdot 2^2 + 80/2 = 60$.

$\square$

7) [10 points] A fixed position [but rotating] camera is placed 0.1 meters away from a straight race track and it is following a race car which is moving at speed of 50 meters per second. [See picture below. You can assume that the car is moving "up" in the picture.] At what speed is the camera rotating [with units radians per second] when it is facing the car with an angle with respect to the line to the closest point of the track [denoted by θ in the figure] of $\pi/4$ radians?



Solution. Let d be as in the picture. So, $d' = 50$. Then, $\tan(\theta) = d/0.1 = 10d$. Hence, taking derivatives, we get $\sec^2(\theta) \cdot \theta' = 10d' = 500$. So, $\theta' = 500/\sec^2(\theta) = 500 \cdot \cos^2(\theta)$. So, when $\theta = \pi/4$, we get $\theta' = 500(\sqrt{2}/2)^2 = 250$. $\square$

8) [10 points] Use differentials/linear approximation to estimate the amount of paint [in cubic centimeters] needed to apply a coat of paint 0.05 cm thick to a sphere of diameter 20 cm.

[**Hint:** The volume of a sphere is $V = 4\pi r^3/3$.]

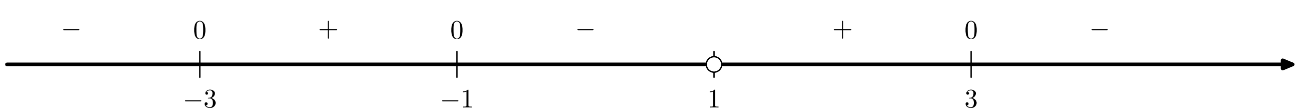
Solution. We have $\mathrm{d}V = 4\pi r^2 \cdot \mathrm{d}r$, i.e., $\Delta V \approx 4\pi r^2 \cdot \Delta r$. Adding a coat of paint will increase the radius by 0.05, i.e., $\Delta r = 0.05$. The variation of the volume ΔV is how much paint was added. So, observing that $r = 20/2 = 10$, we have

$$\Delta V \approx 4\pi 10^2 \cdot 0.05 = 20\pi.$$

So, we'd need 20π cubic centimeters of paint. $\square$

9) [15 points] Sketch the graph of a function $f(x)$ which satisfies all of the following conditions [draw concavities carefully!]:

- domain is all real numbers except 1;
- x-intercepts are -3, 0.25, 1.5, and y-intercept is 1.5;
- $f(-2) = 1.5$, $f(-1) = 3$, $f(3) = 3$, $f(4) = 1.5$;
- $\lim_{x \to -\infty} f(x) = \infty$, $\lim_{x \to \infty} f(x) = 0$, $\lim_{x \to 1} f(x) = -\infty$;
- the sign of the derivative is given by:



- the sign of the second derivative is given by:

Solution. Needs to be drawn by hand... $\square$