1) Compute the following limits. If they do not exist or are infinite, check if the side limits exist.

(a) [5 points] $\lim_{x\to-\infty} \frac{3x^3 - x + 1}{x^3 + x^2}$

Solution.

$$\begin{aligned}\lim_{x\to-\infty} \frac{3x^3 - x + 1}{x^3 + x^2} &= \lim_{x\to-\infty} \frac{x^3}{x^3}\frac{(3 - 1/x^2 + 1/x^3)}{(1 + 1/x)} \\ &= \lim_{x\to-\infty} \frac{3 - 1/x^2 + 1/x^3}{1 + 1/x} = \frac{3}{1} = 3.\end{aligned}$$

□

(b) [5 points] $\lim_{x\to 2} \frac{\frac{1}{x-1} - 1}{x - 2}$

Solution.

$$\begin{aligned}\lim_{x\to 2} \frac{\frac{1}{x-1} - 1}{x - 2} &= \lim_{x\to 2} \frac{1 - (x - 1)}{(x - 1)(x - 2)} \\ &= \lim_{x\to 2} \frac{-(x - 2)}{(x - 1)(x - 2)} \\ &= \lim_{x\to 2} \frac{-1}{(x - 1)} \\ &= \frac{-1}{1} = -1.\end{aligned}$$

□

(c) [10 points] $\lim_{x\to 2} \frac{x^2+x-6}{(x-2)^2}$

Solution.

$$\begin{aligned}\lim_{x\to 2} \frac{x^2+x-6}{(x-2)^2} &= \lim_{x\to 2} \frac{(x-2)(x+3)}{(x-2)^2} \\ &= \lim_{x\to 2} \frac{x+3}{x-2}.\end{aligned}$$

So, analyzing the signs,

$$\begin{aligned}\lim_{x\to 2^+} \frac{x^2+x-6}{(x-2)^2} &= \lim_{x\to 2^+} \frac{x+3}{x-2} \\ &= +\infty\end{aligned}$$

$$\begin{aligned}\lim_{x\to 2^-} \frac{x^2+x-6}{(x-2)^2} &= \lim_{x\to 2^-} \frac{x+3}{x-2} \\ &= -\infty\end{aligned}$$

□

(d) [10 points] $\lim_{x\to -\infty} \sqrt{x^2-3x+1} - 2x$

Solution.

$$\begin{aligned}\lim_{x\to -\infty} \sqrt{x^2-3x+1} - 2x &= \lim_{x\to -\infty} (\sqrt{x^2-3x+1} - 2x)\frac{\sqrt{x^2-3x+1}+2x}{\sqrt{x^2-3x+1}+2x} \\ &= \lim_{x\to -\infty} \frac{x^2-3x+1-4x^2}{\sqrt{x^2-3x+1}+2x} \\ &= \lim_{x\to -\infty} \frac{-3x^2-3x+1}{|x|\sqrt{1-3/x+1/x^2}+2x} \\ &= \lim_{x\to -\infty} \frac{-3x^2-3x+1}{-x\sqrt{1-3/x+1/x^2}+2x} \\ &= \lim_{x\to -\infty} \frac{x^2}{x}\frac{-3-3/x+1/x^2}{-\sqrt{1-3/x+1/x^2}+2} \\ &= \lim_{x\to -\infty} x\,\frac{3+3/x-1/x^2}{\sqrt{1-3/x+1/x^2}-2} \\ &= +\infty\end{aligned}$$

□

2) [15 points] Compute the following derivative (using the formulas, no need for limits):

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x \cdot 2^x - 2x\sqrt{x}}{3x^4 - x^2 + 1}\right)$$

No need to simplify!

Solution.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x \cdot 2^x - 2x\sqrt{x}}{3x^4 - x^2 + 1}\right) &= \frac{\frac{\mathrm{d}}{\mathrm{d}x}(x \cdot 2^x - 2x\sqrt{x}) \cdot (3x^4 - x^2 + 1) - (x \cdot 2^x - 2x\sqrt{x}) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(3x^4 - x^2 + 1)}{(3x^4 - x^2 + 1)^2} \\
&= \frac{\frac{\mathrm{d}}{\mathrm{d}x}(x \cdot 2^x - 2x^{3/2}) \cdot (3x^4 - x^2 + 1) - (x \cdot 2^x - 2x^{3/2}) \cdot (12x^3 - 2x)}{(3x^4 - x^2 + 1)^2} \\
&= \frac{(\frac{\mathrm{d}}{\mathrm{d}x}(x \cdot 2^x) - 3x^{1/2}) \cdot (3x^4 - x^2 + 1) - (x \cdot 2^x - 2x^{3/2}) \cdot (12x^3 - 2x)}{(3x^4 - x^2 + 1)^2} \\
&= \frac{((2^x + x2^x \ln(2)) - 3x^{1/2}) \cdot (3x^4 - x^2 + 1) - (x \cdot 2^x - 2x^{3/2}) \cdot (12x^3 - 2x)}{(3x^4 - x^2 + 1)^2}
\end{aligned}$$

□

3) [20 points] Let

$$f(x) = \begin{cases} x^3, & \text{if } x < 0, \\ x^2, & \text{if } x \geq 0. \end{cases}$$

Is $f(x)$ continuous at $x = 0$? Is it differentiable? If so, compute $f'(0)$. [Show work!]

Solution. We have:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0,$$

and

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^3 = 0.$$

So,

$$\lim_{x\to 0} f(x) = 0 = f(0),$$

and therefore f is continuous at $x = 0$.

Now,

$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{f(h) - 0}{h} = \lim_{h\to 0^+} \frac{h^2}{h} = \lim_{h\to 0^+} h = 0.$$

On the other hand,

$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{f(h) - 0}{h} = \lim_{h\to 0^3} \frac{h^3}{h} = \lim_{h\to 0^-} h^2 = 0.$$

So, f is differentiable and,

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = 0.$$

□

4) [15 points] Let $f(x) = x\mathrm{e}^x - x^6 + 1$. Show that $f(x)$ has at least two zeros [i.e., there are $a, b \in \mathbb{R}$, with $a \neq b$, such that $f(a) = f(b) = 0$].
Hint: I am not asking you to *find* these zeros, just show their existence. Also, e is approximately 2.72.

Solution. We have: $f(0) = 1$

$$f(-1) = -\mathrm{e}^{-1} - 1 + 1 = -\frac{1}{\mathrm{e}} < 0.$$

Hence, by the *Intermediate Value Theorem*, we have that $f(x)$ has a zero between -1 and 0.

Also,

$$f(2) = 2\mathrm{e}^2 - 2^6 + 1 < 2 \cdot 3^2 - 64 + 1 = 18 - 63 = -45 < 0.$$

Hence, by the *Intermediate Value Theorem*, we have that $f(x)$ has a zero between 0 and 2.

□

5) [20 points] The graph in the middle is the graph of $y = f'(x)$ for some function f such that $f(0) = 0$. Sketch $y = f(x)$ in the grid *above* the given graph and beneath it sketch the graph of $y = f''(x)$.