Midterm (Take Home) – Solution

M552 – Abstract Algebra

1. Let R be a *local ring*, i.e., a commutative ring with 1 with a unique maximal ideal, say I, and let M be a *finitely generated* R-modulo.

(a) [10 points] If N is a submodule of M and $M = N + (I \cdot M)$, then $M = N$.

[**Hint:** Last semester I proved *Nakayma's Lemma* for ideals. The same proof works for [finitely generated] modules. [See Proposition 16.1 on pg. 751 of Dummit and Foote.] Use it here.]

Proof. Since R is local, we have that its Jacobson radical is I.

Now, since M is finitely generated, then so is M/N. [Generated by the classes of the generators of M.]

So, $M/N = (N+IM)/N = (IM)/N = I(M/N)$. [More formally, let $\alpha = m+N \in M/N$, with $m \in M$. But, $M = N + (I \cdot M)$, and so there are $n' \in N$, $x' \in I$, and $m' \in M$, such that $m = n' + x'm'$. Hence, $\alpha = (n' + x'm') + N = x'm' + N$. Thus, $\alpha \in I(M/N)$, and hence $M/N = I(M/N)$ [since the other inclusion is trivial].] By Nakayama's Lemma, we have that $M/N = 0$, i.e., $M = N$.

□

(b) [30 points] Suppose further that M is *projective* [still with the same hypothesis as above]. Prove that M is free.

[**Hints:** Look at $M/(I \cdot M)$ to find your candidate for a basis. Use (a) to prove it generates M. Then let F be a free module with the rank you are guessing to be the rank of M and use (a) to show that the natural map $\phi : F \to M$ is an isomorphism.]

Proof. We have that $M/(I \cdot M) \cong (R/I) \otimes_R M$. Since I is a maximal ideal, we have that R/I is a field, and hence, $M/(I \cdot M)$ is a vector space. Let $\{\bar{x}_1, \ldots, \bar{x}_n\}$ be a basis of this vector space. [Note, if $M/IM = 0$, then $M = IM$, and so, by Nakayama's Lemma, $M = 0$.]

Let $\{x_1, \ldots, x_n\} \subseteq M$ such that $x_i + I \cdot M = \bar{x}_i$. Let N be submodule generated by this set. Then, clearly, $M = N + I \cdot M$, and hence $M = N$ by (a). So, we have that $\{x_1, \ldots, x_n\}$ generates M.

[Now we will basically reprove the fact that if M is projective, than it's a direct summand of a free module. We do it to make the other summand explicit.] Let F be a free R-module with basis $\{e_1, \ldots, e_n\}$ and consider $\phi : F \to M$ defined by

$\phi(e_i) = x_i$. Then, ϕ is clearly onto. Let $K \stackrel{\text{def}}{=} \ker\phi$. This gives us the short exact sequence:

$$0 \longrightarrow K \xrightarrow{\text{incl.}} F \xrightarrow{\phi} M \longrightarrow 0.$$

Since M is projective, this sequence splits. So, $F = M \otimes K$. Now, suppose that $\alpha = r_1e_1 + \cdots + r_ne_n \in K$. Then, $\phi(\alpha) = r_1x_1 + \cdots + r_nx_n = 0$ in M. So, $\bar{r}_1 \otimes \bar{x}_1 + \cdots + \bar{r}_n \otimes \bar{x}_n = 0$ in $M/(I \cdot M)$ [seen here as vector space over R/I]. Hence, since $\{\bar{x}_1, \ldots, \bar{x}_n\}$ is an R/I-basis of $M/(I \cdot M)$, we have that $\bar{r}_i = 0$, i.e., $r_i \in I$. Therefore, $K \subseteq I \cdot F$.

But then, $F = M \otimes K = M + I \cdot F$ [where we identify M with its copy in F via the splitting homomorphism]. So, by (a), we have that $F = M$, and hence M is free.

Alternative proof: Here is a proof that a couple of you found. It's actually nicer than the one above [to which I led you through my hint].

Since, M is finitely generated, let $\{e_1, \ldots, e_n\}$ be a generating set with *minimal size*. [We may assume $M \neq 0$.] Let F and ϕ be as above [and deduce that $F = \ker\phi \oplus M$].

Suppose that $r_1x_1 + \cdots + r_nx_n \in K = \ker\phi$. Then, $\phi(r_1x_1 + \cdots + r_nx_n) = r_1e_1 + \cdots r_ne_n = 0$. Thus, if any $r_i \in R - I$, then it is a unit [since R is local]. But, if, without loss of generality, $r_1 \in R^\times$, then, $e_1 = -r_1^{-1}r_2e_2 - \cdots - r_1^{-1}r_ne_n$, which would mean that $\{e_2, \ldots, e_n\}$ generates M, contradicting our minimality assumption. So, $r_i \in I$ for every i. Hence, $K \subseteq IM$, and, as above [in the previous proof], we get $F \cong M$.

$\square$