Midterm (In Class) - Solutions

M552 – Abstract Algebra

1. Modules:

(a) [10 points] Give an example of an *injective* homomorphism of [left] R-modules $\phi : N \to M$ such that $1 \otimes \phi : L \otimes_R N \to L \otimes_R M$ is *not* an injection for some [right] R-module L. [You do *not* have to repeat computations of tensor products that were done in class, in HW, or in Dummit and Foote.]

Proof. [This was done in class.] Let $R = \mathbb{Z}$, $N = \mathbb{Z}$, $M = \mathbb{Q}$, and ϕ be the natural inclusion. Now, take $L = \mathbb{Z}/p\mathbb{Z}$. Then, $\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z}$, and $\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, so $1 \otimes \phi$ is not an inclusion. [So, $\mathbb{Q}$ is not *flat*.]

□

(b) [20 points] Let R be a commutative ring with 1. Show that if every R module is free, then R is a PID.

Proof. Let I be a non-zero ideal of R [not necessarily principal]. Then I is an R-submodule of R, and hence it's free. So, suppose that e_1 and e_2 are two distinct elements of a basis of I, i.e., that the rank of I is greater than one. Then, clearly $e_1, e_2 \neq 0$, while $e_2 e_1 - e_1 e_2 = 0$ [since R is commutative], which is a contradiction, since $\{e_1, e_2\}$ should be linearly independent.

Thus, I has rank one, if $\{a\}$ is a basis, clearly $I = Ra = (a)$, and I is principal.

Now, we just need to show that R is an integral domain. So, let $I = (a)$ be an ideal, with $a \neq 0$. By the above, $ra \mapsto r$ is an isomorphism of R-modules. If there exists $b \in R - \{0\}$ such that $ba = 0$, then, then, on one hand $ba \mapsto b \neq 0$, while on the other other we get $ba = 0 \mapsto 0$, giving us a contradiction. So, R is a domain [since it's already commutative with 1].

Note: If fact, R must be a field! Let $I \neq R$ be an ideal and $M = R/I$. Then, R/I is free, which means that the annihilator of R/I is zero. [Just zero annihilates a basis element.] But I annihilates R/I, so we have $I = 0$. So, the ideals of R are I and (0), and so [since R is commutative with 1], R must be a field [and hence a PID].

A better problem would have been to prove that if every *submodule of a free R-module* is free, than R is a PID. The proof above shows this, but I thought I would give a slightly easier problem.

Note that, together, these give the converse for two statements:

- If R is a field, then every R-module is free.
- If R is a PID, then very submodule of a free R-module is free.

□

2. Linear algebra:

(a) [10 points] Let $B \in \mathrm{M}_n(\mathbb{C})$ [$n \times n$ matrices with entries in $\mathbb{C}$] be a *block diagonal* matrix. Prove that B is diagonalizable if, and only if, each block is. [You can use the algebra of block matrices without proof.]

Proof. Let B_i be a block of B and P_i an invertible matrix such that $P_i^{-1} B_i P_i$ is in Jordan canonical form, say J_i. Then, the inverse of the block diagonal matrix P with P_i as blocks is a block diagonal matrix with P_i^{-1} as blocks. Then, $P^{-1}BP$ is a block diagonal matrix with J_i as blocks. This is a Jordan canonical form of the matrix B [since each J_i is made of Jordan blocks, and so B is made of Jordan blocks].

Then, clearly B is diagonalizable if, and only if, each J_i is diagonal, i.e., each block is diagonalizable.

Note: One could also prove that $m_B = \mathrm{lcm}(m_{B_1}, \ldots, m_{B_n})$, again using simple block diagonal matrix algebra. This would say that m_B has only simple roots if, and only if, each m_{B_i} does.

□

(b) [20 points] Let $A, B \in \mathrm{M}_n(\mathbb{C})$ be two *diagonalizable* matrices. Prove that there is [a *single*] $P \in \mathrm{GL}_n(\mathbb{C})$ [i.e., an invertible matrix] such that *both* $P^{-1}AP$ and $P^{-1}BP$ are diagonal [so they are *simultaneously diagonalizable*] if, and only if, $AB = BA$. [**Hint:** Look at eigenspaces.]

Proof. Suppose $P^{-1}AP$ and $P^{-1}BP$ are both diagonal. Then, they commute, giving us $P^{-1}ABP = P^{-1}APP^{-1}BP = P^{-1}BPP^{-1}AP = P^{-1}BAP$. Since P is invertible, this gives us $AB = BA$.

Assume now $AB = BA$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A and $V_{A,\lambda}$ its corresponding eigenspace. Then, if $v \in V_{A,\lambda}$, we have $AB(v) = BA(v) = B(\lambda v) = \lambda(Bv)$. So, $B(v) \in V_{A,\lambda}$. Hence, $V_{A,\lambda}$ is invariant under B. Hence, the matrix associated to B with respect to the basis of eigenvectors of A [which exists since A is diagonalizable] is a block matrix, each block corresponding to an eigenspace [or eigenvalue].

By part (a), since B is diagonalizable, each block is. So, there is change of basis of $V_{A,\lambda}$ such that the corresponding block is diagonal, i.e., the new basis is formed of eigenvalues of B. Note that this new basis is also formed of eigenvectors of A, since it is a basis for an eigenspace of A.

Therefore, this new basis is made of eigenvalues of A and B *simultaneously*, and so the transition matrix from the canonical basis to this new one diagonalizes both matrices at the same time.

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