May 2nd, 2007

Math 456

Luís Finotti
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Name: ....

Student ID (last 5 digits): XXX-X................

Final

Staple to the statements [this document] your solutions in the proper order. [Make sure you are not missing any problem.] They have to be clearly written and organized. Start every new problem in a different sheet.

Check that no pages of the staments [this document] are missing. It has 8 questions and 3 printed pages [including this one], all of which should be turned in with the solutions.

You can use your notes and textbook, but you cannot talk or say *anything at all* about this exam to *anyone*. If you do, you will get an **F** in this course.

Show all work! Even correct answers without work may result in point deductions. **Points will be taken from messy solutions.**

You have until 12:15pm on 05/02 (Wednesday) to turn in your solutions in my office. I strongly recommend that you plan to turn it in early. [Don't leave it for the last minute!!] If you are late, you will be penalized. [If you foresee any problems, please contact me ASAP.]

Good luck!

Question	Max. Points	Score
1	11	
2	11	
3	11	
4	11	
5	11	
6	15	
7	15	
8	15	
Total	100	

1) Let $\alpha_1 \stackrel{\text{def}}{=} 8 - 8\mathrm{i}$, $\alpha_2 \stackrel{\text{def}}{=} 10 + 15\mathrm{i}$, $\beta \stackrel{\text{def}}{=} 2 - 3\mathrm{i}$, and let $I \stackrel{\text{def}}{=} (\beta)$ be the principal ideal of $\mathbb{Z}[\mathrm{i}]$ generated by β.

(a) Compute the quotient and remainders of the divisions of α_1 and α_2 by β?

(b) Is $\alpha_1 \equiv \alpha_2 \pmod{I}$?

2) Let $\zeta_{11} \stackrel{\text{def}}{=} \mathrm{e}^{2\pi\mathrm{i}/11}$. How many intermediate fields does the extension $\mathbb{Q}[\zeta_{11}]/\mathbb{Q}$ have [including $\mathbb{Q}$ and $\mathbb{Q}[\zeta_{11}]$]? What are their degrees over $\mathbb{Q}$? [You do **not** have to find them, just count them and give their degrees.]

3) Let R be a ring [which you can assume is commutative with identity, but it is not necessary] and $a \in R$. Let $\phi : R \to R'$ be a homomorphism such that $a \in \ker \phi$. Prove that the map $\psi : R/(a) \to R'$, defined by $\psi(b + (a)) \stackrel{\text{def}}{=} \phi(b)$ gives a *well-defined* [you *have* to prove that it is well-defined] ring homomorphism.

4) Prove that if F is a field and $F[[x]]$ represents *formal power series* over F [as in the second extra-credit problem], then *all non-zero* ideals of $F[[x]]$ are of the form (x^n) where n is a non-negative integer. [**Hint:** You can use any fact in the statement of the extra-credit problem.]

5) *Construct* a field with 8 elements. [**Hint:** Extend some known field.]

6) Let F be a field of characteristic $p \neq 0$, for which the polynomial $f(x) \stackrel{\text{def}}{=} x^p - x - a \in F[x]$ is irreducible. Let α be a root of $f(x)$ [in some extension of F].

(a) Prove that $\alpha + 1$ is also a root of $f(x)$.

(b) Prove that $F[\alpha]$ is the splitting field of $f(x)$ over F. [**Hint:** Use (a) to find all roots of f.]

(c) Prove that $G(F[\alpha]/F)$ is *cyclic*.

7) Let $K \stackrel{\text{def}}{=} \mathbb{Q}[\sqrt[4]{2}, \mathrm{i}]$.

(a) Find $[K : \mathbb{Q}]$.

(b) Give a $\mathbb{Q}$-basis for K [as a vector space over $\mathbb{Q}$].

(c) Prove that $K/\mathbb{Q}$ is Galois.

(d) If $\sigma \in G(K/\mathbb{Q})$, then what are the possible values of $\sigma(\sqrt[4]{2})$ and $\sigma(\mathrm{i})$?

8) In this problem we will show that if R is commutative ring with identity, and $a \in R$ is such that $a^n = 0$ for some positive integer n, then a is in every maximal ideal of R. [Note that if $a \neq 0$, then R is **not** an integral domain!]

(a) Let I be an ideal and $a \in R$. Prove that

$$(I, a) \stackrel{\text{def}}{=} \{x + ra \ : \ x \in I \text{ and } r \in R\}$$

is an ideal of R that contains I and a.

(b) Prove that if M is a *maximal* ideal and $a^n = 0$ [and you can assume $a^{n-1} \neq 0$] for some positive integer n, with $a \notin M$ [to later derive a contradiction], then $a^{n-1} \in M$. [**Hint:** Start by proving that $1_R \in (M, a)$.]

(c) Prove that since $a^{n-1} \in M$, we actually have $a \in M$ [which is then a contradiction to the fact that $a \notin M$].