1) Let $\mathcal{F}$ and $\mathcal{G}$ be non-empty families of sets. Prove that if every element of $\mathcal{F}$ is a subset of *every* element of $\mathcal{G}$, then $\bigcup\mathcal{F} \subseteq \bigcap\mathcal{G}$.

Proof. Let $x \in \bigcup\mathcal{F}$. Then, there is $A \in \mathcal{F}$ such that $x \in A$.
Now, let $B \in \mathcal{G}$. Since $A \in \mathcal{F}$, by assumption we have that $A \subseteq B$. Then, since $x \in A$ and $A \subseteq B$, we have that $x \in B$.
Since $B \in \mathcal{G}$ was arbitrary, we have that $x \in B$ for all $B \in \mathcal{G}$, i.e., $x \in \bigcap\mathcal{G}$. $\square$

2) Prove that for every integer n, n^3 is even if and only if n is even.
[**Note:** This was a HW problem.]

Proof. ($\rightarrow$): Suppose that n is even. Then, $n = 2k$ for some integer k. Then, $n^3 = (2k)^3 = 8k^3 = 2 \cdot (4k^3)$. Hence, since $4k^3 \in \mathbb{Z}$, we have that n is even.

($\leftarrow$): Now assume that n is odd. [Let's prove the contrapositive.] Then, $n = 2k+1$ for some integer k. Then, $n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2 \cdot (4k^3 + 6k^2 + 3k) + 1$. Since $4k^3 + 6k^2 + 3k \in \mathbb{Z}$, we have that n^3 is odd. $\square$

3) Let $\mathcal{F}$ and $\mathcal{G}$ be non-empty families of sets. Prove that $\bigcup(\mathcal{F} \cup \mathcal{G}) \subseteq (\bigcup\mathcal{F}) \cup (\bigcup\mathcal{G})$.

Proof. Let $x \in \bigcup(\mathcal{F} \cup \mathcal{G})$. Then, $x \in A$ for some $A \in \mathcal{F} \cup \mathcal{G}$, so $A \in \mathcal{F}$ or $A \in \mathcal{G}$.

Case 1: Suppose that $A \in \mathcal{F}$. Since $x \in A$, we have that $x \in \bigcup\mathcal{F}$. Thus, $x \in (\bigcup\mathcal{F}) \cup (\bigcup\mathcal{G})$.

Case 2: Suppose that $A \in \mathcal{G}$. Since $x \in A$, we have that $x \in \bigcup\mathcal{G}$. Thus, $x \in (\bigcup\mathcal{F}) \cup (\bigcup\mathcal{G})$.
$\square$

4) Let U be a set. Prove that there is a unique $A \in \mathscr{P}(U)$ such that for all $B \in \mathscr{P}(U)$ we have $A \cup B = A$.

Proof. (Existence): Take $A = U$. Since $U \subseteq U$, we have that $U \in \mathscr{P}(U)$. Also, for all $B \in \mathscr{P}(U)$, we have that $B \subseteq U$, and hence $A \cup B = U \cup B = U$. [There is no need to prove this, but here it goes: if $x \in U$, then $x \in U \cup B$. If $x \in U \cup B$, then $x \in B$ or $x \in U$. But since $B \subseteq U$, we have in either case that $x \in U$.]

(Uniqueness): Suppose that there is some A' such that for all $B \in \mathscr{P}(U)$ we have $A' \cup B = A'$. Then, we can take $B = U$, so $A' \cup U = A'$. But, by the above, $A' \cup U = U$, and hence $A' = U$. □

5) Let A, B. C, and D be sets. Prove that if $A \times B$ and $C \times D$ are disjoint, then either A and C or B and D are disjoint.
[**Note:** This was a HW problem.]

Proof. Suppose that neither A and C, nor B and D are disjoint. [We prove the contraceptive.] Then, there is come $x \in A \cap C$ and some $y \in B \cap D$. Since $x \in A$ and $y \in B$, we have that $(x, y) \in A \times B$. Since also $x \in C$ and $y \in D$, we have that $(x, y) \in C \times D$. Therefore $(x, y) \in (A \times B) \cap (C \times D)$, and hence $A \times C$ and $B \times D$ are not disjoint. □