1) [25 points] Let n be a [fixed] positive integer and define:

$$R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid \ n \mid (a - b)\}.$$

Prove that R is an equivalence relation on $\mathbb{Z}$. [Remember, $x \mid y$ if there is $k \in \mathbb{Z}$ such that $y = x \cdot k$.]
[I mentioned this result in class and quickly said, but did not write, all the steps.]
Partial credit: If you can't do this or are stuck, I will give half credit [12 points] for the definitions of symmetric, reflexive and transitive relations.

Proof. [Reflexive:] Let $a \in \mathbb{Z}$. Since $n \mid 0 = a - a$, [as $0 = n \cdot 0$ and $0 \in \mathbb{Z}$], we have aRa.

[Symmetric:] Suppose that aRb. Then, $n \mid (a - b)$, i.e., $(a - b) = n \cdot k$ for some $k \in \mathbb{Z}$. Then, $(b - a) = n \cdot (-k)$. Since $-k \in \mathbb{Z}$ [as $k \in \mathbb{Z}$], we have that $n \mid (b - a)$. Thus, bRa.

[Transitive:] Suppose that aRb and bRc. Then, $n \mid (a-b)$ and $n \mid (b-c)$. So, there are $k, l \in \mathbb{Z}$ such that $(a-b) = n \cdot k$ and $(b-c) = n \cdot l$. Thus, $(a-c) = (a-b)+(b-c) = n \cdot k + n \cdot l = n \cdot (k+l)$. Since $k + l \in \mathbb{Z}$ [as $k, l \in \mathbb{Z}$], we have that $n \mid (a - c)$, and so aRc. $\square$

2) [25 points] Let $f : A \to B$ and $g : B \to C$. Prove that if $g \circ f$ is one-to-one, then f is one-to-one.
[This was a homework problem.]
Partial credit: If you can't do this or are stuck, I will give some credit [10 points] for the definition of one-to-one.

Proof. Suppose that $f(a) = f(a')$. [We need $a = a'$.] Then, $g(f(a)) = g(f(a'))$, i.e., $g \circ f(a) = g \circ f(a')$. Since $g \circ f$ is one-to-one, we have that $a = a'$. $\square$

3) [25 points] Let $f : A \to B$ and $g : B \to C$. Prove that if f is onto and g is not one-to-one, then $g \circ f$ is not one-to-one.
[This was a homework problem.]
Partial credit: If you can't do this or are stuck, I will give some credit [12 points] for the definition of onto and the negation of the definition of one-to-one.

Proof. Since g is not one-to-one, there are $b, b' \in B$, with $b \neq b'$, such that $g(b) = g(b')$. Since f is onto, there are $a, a' \in A$ such that $f(a) = b$ and $f(a') = b'$. Since $b \neq b'$, we have that $a \neq a'$. Also, we now have $g(f(a)) = g(f(a'))$ with $a \neq a'$, so $g \circ f$ is not one-to-one. $\square$

4) [25 points] Let $f : A \to B$ and $g : B \to A$. Prove that if f is onto and $g \circ f = i_A$, then $f \circ g = i_B$. [Note that this means that $g = f^{-1}$.]
[This was done in a video.]

Proof. Let $b \in B$. [Need to show that $f \circ g(b) = b$.] Since f is onto, there is $a \in A$ such that $f(a) = b$. So, $g(f(a)) = g(b)$. But, since $g \circ f = i_A$, we have that $g(f(a)) = a$. So, $g(b) = a$ and then $f \circ g(b) = f(g(b)) = f(a) = b$. $\square$

Alternative Proof. Since $g \circ f = i_A$, we have that f in one-to-one. So, since it is also onto, we have that $f^{-1} : B \to A$. So,

$$f \circ g = f \circ g \circ i_B = f \circ g \circ (f \circ f^{-1}) = f \circ (g \circ f) \circ f^{-1} = f \circ i_A \circ f^{-1} = f \circ f^{-1} = i_B.$$

$\square$